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The incompressible Navier–Stokes limit from the discrete-velocity BGK Boltzmann equation

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Abstract

In this paper, we extend the Bardos–Golse–Levermore program (Bardos *et al* 1993 *Commun. Pure Appl. Math.* **46** 667–753) to prove that a local weak solution to the *d*-dimensional incompressible Navier–Stokes equations ($d \geqslant 2$) can be constructed by taking the hydrodynamic limit of a discrete-velocity Boltzmann equation with a simplified Bhatnagar–Gross–Krook collision operator. Moreover, in the case when the dimension is $d = 2, 3$, we characterise the combinations of finitely many particle velocities and probabilities that lead to the incompressible Navier–Stokes equations in the hydrodynamic limit. Numerical computations conducted in two-dimensional indicate that in the case of the simplest velocity lattice (D2Q9), the rate with which this hydrodynamic limit is achieved is of order $\mathcal{O}(\varepsilon^2)$, where $\varepsilon \to 0$ is the Knudsen number. For

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the future investigations, it is worth considering if the hydrodynamic limit of the discrete-velocity Boltzmann equation can be also rigorously justified in the presence of non-trivial boundary conditions.

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1. Introduction

The lattice Boltzmann method (LBM for short), originating from lattice gas automata, is a successful and promising numerical scheme for simulating fluid flows. Different from conventional schemes which are based on macroscopic continuum equations, the LBM is based on microscopic kinetic equations. The approach of using microscopic kinetic equations has advantages over other computational fluid dynamics methods in the sense of easy implementation of boundary conditions and parallel algorithms. The underlying reason for the effectiveness of the LBM is because of the fact that macroscopic fluid phenomena are collective behaviours of microscopic interactions between particles. The LBM is characterised by the numerical implementation of the lattice Boltzmann equation (LBE for short)

$$f_i\left(x + c_i\,\Delta x, t + \Delta t\right) = f_i\left(x, t\right) + \Omega_i\left(x, t\right), \quad i = 0, 1, \ldots, M, \tag{1}$$

where M represents the number of non-zero particle velocities that we want to consider, c_i represents the ith local particle velocity, f_i represents the particle velocity distribution function along c_i and Ω_i represents the collision operator which characterises the rate of change of f_i resulting from collisions. There are many different choices for the collision operator Ω_i in (1). Among which, the one that is most commonly used for Navier–Stokes simulations is the lattice Bhatnagar–Gross–Krook (BGK) collision operator [5]

$$\Omega_i\left(f\right) = \frac{\Delta t}{\tau_{\mathrm{R}}}\left(f_{i,\mathrm{eq}} - f_i\right), \quad f_{i,\mathrm{eq}} = w_i\,\varrho\left(1 + 3c_i \cdot U + \frac{9}{2}\left(c_i \cdot U\right)^2 - \frac{3}{2}|U|^2\right) \tag{2}$$

where τ_{R} is the relaxation time, w_i represents certain weight for velocity c_i,

$$\varrho = \sum_i f_i \quad \text{and} \quad \varrho U = \sum_i c_i f_i.$$

In particular, ϱ is the macroscopic fluid density and U is the macroscopic fluid velocity. The LBE (1) with collision operator (2) is usually referred to the 'lattice BGK Boltzmann equation', see e.g. [26]. Throughout this paper, we use $d \in \mathbf{N}$ to denote the space dimension and we work with the case where the dimension $d \geqslant 2$. The $f_{i,\mathrm{eq}}$ term in the lattice BGK collision operator (2) is derived from the expansion of the Maxwell–Boltzmann equilibrium distribution

$$\mathcal{M}_f\left(x, v, t\right) = \frac{\varrho}{\left(2\pi\theta\right)^{d/2}}\exp\left(-\frac{|v - U|^2}{2\theta}\right) \tag{3}$$

with respect to the velocity variable v in Hermite polynomials up to the third moment, i.e. polynomials in v of power 2; see e.g. [16, chapter 3]. In the standard BGK Boltzmann equation, the factor θ in (3) is the macroscopic fluid temperature defined by

$$\varrho U^2 + \mathrm{d}\varrho\theta = \int_{\mathbf{R}^d} f(x,v,t)\,|v|^2\,\mathrm{d}v.$$

In addition, the $f_{i,\mathrm{eq}}$ term in (2) is obtained under the isothermal assumption that $\theta = 1$.

It is well-known that the Navier–Stokes equations can be derived formally from the LBE (1) by the Taylor expansion and the Chapman–Enskog expansion, see, e.g. [8, 18, 30]. This is the mathematical foundation for the use of the lattice Boltzmann equation (1) to model fluid dynamics. Lallemand *et al* [18] provided a comprehensive overview of the LBM and of the mathematical theory behind this method. Simonis and Krause [28] introduced the concept of 'limit consistency' to formally prove the consistency of the LBM schemes based on an analysis of discretisations satisfying certain limiting properties. Motivated by the Taylor expansion, Guo *et al* [12] introduced the 'unified preserving property' to consistent numerical schemes for the LBE, so that the asymptotic orders of kinetic schemes can be assessed by employing the modified equation approach and Chapman–Enskog analysis. Kummer and Simonis [17] proved the nonuniqueness of solutions to lattice Boltzmann formulations obtained with multi-step finite difference schemes, which indicates the existence of equivalence classes, defined by generalised matrix similarity, for moment matrices used in defining a lattice Boltzmann scheme. The convergence of the LBM to Navier–Stokes flows on periodic and bounded domains was given by Junk and Yang [14]. Wissocq and Sagaut [30] conducted a detailed exploration of numerical errors that arise in the implementation of the LBE with an acoustic scaling, by first studying the hydrodynamic limit in the discrete setting and then investigating the numerical consistency with respect to the compressible Navier–Stokes system. Overall, most of the results on the LBM are only formal and, from the perspective of mathematical analysis, are not rigorous since not only it is not known how (1) relates to the discretisation of the Navier–Stokes equations, but also the justification of how the continuous counterpart of (1) relates to the Navier–Stokes equations is unclear. In order to justify the relation between the continuous counterpart of (1) and the Navier–Stokes equations in the most rigorous way, one should take into account of the Knudsen number for the continuous counterpart of LBE and investigate its hydrodynamic limit as the Knudsen number converges to zero.

The study of the hydrodynamic limit from the Boltzmann equation was initiated by the work of Bardos *et al* [4], where they derived Leray solutions to the incompressible Navier–Stokes equations from DiPerna's–Lions' renormalised solutions [9] of the Boltzmann equation with Grad's cutoff kernel. Since then, the hydrodynamic limit of the Boltzmann equation has become one of the major research topics in fluid mechanics, even until today. Along with the philosophy of Bardos *et al* [4], BGL for short, many significant improvements were established subsequently. For example, Lions and Masmoudi [21] extended the work of Bardos *et al* [4] to a more general time-continuous case. Later on, Golse and Saint-Raymond [10] proved the convergence of DiPerna–Lions' renormalised solutions [9] to Leray solutions in the case for hard cutoff potentials. Same convergence for the case of soft potentials was established by Levermore and Masmoudi [20]. Furthermore, Arsenio [1] extended this convergence to the case for non-cutoff potentials. Saint-Raymond [27] considered the Boltzmann equation with

BGK collision operator and showed that solutions to the Boltzmann equation with BGK collision operator that are fluctuations near the Maxwellian converge in hydrodynamic limit to Leray solutions of the incompressible Navier–Stokes–Fourier equations. It should be remarked that for all these fascinating results that are mentioned above, the solvability of the Boltzmann equation is proved using an entropy method approach. In order to obtain solutions with better regularity than Leray solutions in the hydrodynamic limit, Jiang *et al* [13] employed an energy method approach to solve the Boltzmann equation in cases for both non-cutoff and cutoff collision operators. They derived a global energy estimate for the Boltzmann equation which holds when the initial data is sufficiently small. This global energy estimate guarantees the existence of a global-in-time solution to the Boltzmann equation with Sobolev $H^N (N \geqslant 3)$ regularity in spatial variable x when the initial data is sufficiently small. Hence, by taking the hydrodynamic limit for this family of global solutions, a global classical solution to the incompressible Navier–Stokes–Fourier equations can be obtained provided that the initial fluid velocity is sufficiently small.

Apart from the BGL program, another method that can rigorously justify the hydrodynamic limit of a Boltzmann-type equation relies on Hilbert expansions where one constructs the solution f_ε to the Boltzmann equation in the form of $f_\varepsilon = \mu + \varepsilon \sqrt{\mu}(f_1 + \varepsilon f_2 + \ldots + \varepsilon^{n-1} f_n + \varepsilon^n g_\varepsilon)$ in which $f_1, f_2, \ldots, f_n$ are given by solutions of the prescribed macroscopic fluid equations and g_ε is the remainder term. Hence, for this approach to work, the solvability of the prescribed fluid equations has to be established in advance. This philosophy is very different from the BGL program, where one does not need to assume any *a priori* information about the macroscopic fluid equations. It derives the fluid equations from solutions of the Boltzmann equation, and consequently gives solutions to the fluid equations by taking the limit. In other words, the Hilbert expansion establishes a link between the solution of the Boltzmann equation and the solutions of the fluid equations, provided that the fluid equations are solvable, whereas the BGL program proves the solvability of the macroscopic fluid equations by taking limits of suitable moments for the solution of the Boltzmann equation. However, the Hilbert expansion is an approach that would produce an explicit convergence rate, see e.g. [22], due to the ansatz in which f_i has a coefficient ε^i and g_ε has a coefficient ε^{n+1} in the Hilbert expansion. This convergence rate is something that is hard to obtain from the BGL program. The purpose of this paper is to give a rigorous justification for the hydrodynamic limit of a velocity-discretised Boltzmann equation with simplified BGK collision operator by following the BGL program. It should be emphasised that Junk and Yong [15] studied analogous questions in two-dimensional (2D) using an approach based on the Hilbert expansion. Comparing to what they established, our work develops a more comprehensive theory applicable to different spatial dimensions and to more general lattice structures. More significantly, our work also presents a detailed proof of the local well-posedness of the velocity-discretised Boltzmann equation, whereas in [15] this aspect was skipped.

Next, we introduce our model, i.e. the discrete-velocity BGK (DVBGK for short) Boltzmann equation, in an explicit way. We shall start by introducing the lattice structure with which the macroscopic fluid equations can be derived in hydrodynamic limit. Let $n \in \mathbf{N}$ where n denotes the number of non-zero velocities that we want to consider in a lattice structure. Let $\mathcal{V} := \{v_0 = 0, v_1, \ldots, v_n\} \subset \mathbf{R}^d$ be a set of velocities for particles such that $v_i \neq v_j$ for any $i, j \in \{0, 1, \ldots, n\}$ satisfying $i \neq j$. Let $w = (w_0, w_1, \ldots, w_n) \in \mathbf{R}_+^{n+1}$ be a weight that we are going to impose on $\mathcal{V}$ where $\mathbf{R}_+ := \{a \in \mathbf{R} \mid a > 0\}$. We call the pair $(\mathcal{V}, w)$ a lattice.

Definition 1. We call a lattice $(\mathcal{V}, w)$ to be an isotropic lattice associated with the speed of sound $c_{\mathrm{s}} > 0$ if they satisfy

$$
\begin{cases}
\displaystyle\sum_{i=0}^{n} w_i = 1, \\[2mm]
\displaystyle\sum_{i=0}^{n} w_i v_{i,\alpha} = 0, \\[2mm]
\displaystyle\sum_{i=0}^{n} w_i v_{i,\alpha} v_{i,\beta} = c_{\mathrm{s}}^2 \delta_{\alpha\beta}, \\[2mm]
\displaystyle\sum_{i=0}^{n} w_i v_{i,\alpha} v_{i,\beta} v_{i,\gamma} = 0, \\[2mm]
\displaystyle\sum_{i=0}^{n} w_i v_{i,\alpha} v_{i,\beta} v_{i,\gamma} v_{i,\zeta} = c_{\mathrm{s}}^4 \left(\delta_{\alpha\beta} \delta_{\gamma\zeta} + \delta_{\alpha\gamma} \delta_{\beta\zeta} + \delta_{\alpha\zeta} \delta_{\beta\gamma} \right),
\end{cases}
\tag{4}
$$

where the notation $v_{i,\eta}$ ($1 \leqslant \eta \leqslant d$) represents the η-component of particle velocity v_i.

The notion of an 'isotropic lattice' follows from the standard terminology describing lattice symmetries in the classical lattice Boltzmann models, see, e.g. [31]. Let $\varepsilon > 0$ be the Knudsen number and $(\mathcal{V}, w)$ be an isotropic lattice. We define the DVBGK Boltzmann equation on $(\mathcal{V}, w)$ in $\mathbf{R}^d$ to be the vector system

$$
\varepsilon \partial_t g^\varepsilon + v \cdot \nabla_x g^\varepsilon = \frac{1}{\varepsilon \nu} \left(g_{\mathrm{eq}}^\varepsilon - g^\varepsilon \right), \quad g^\varepsilon \big|_{t=0} = g^{0,\varepsilon},
\tag{5}
$$

where ν denotes the relaxation time,

$$
g^\varepsilon := \left(g_0^\varepsilon, g_1^\varepsilon, \ldots, g_n^\varepsilon \right), \quad g_{\mathrm{eq}}^\varepsilon := \left(g_{0,\mathrm{eq}}^\varepsilon, g_{1,\mathrm{eq}}^\varepsilon, \ldots, g_{n,\mathrm{eq}}^\varepsilon \right), \quad g^{0,\varepsilon} := \left(g_0^{0,\varepsilon}, g_1^{0,\varepsilon}, \ldots, g_n^{0,\varepsilon} \right)
$$

and

$$
v := \left(v_0, v_1, \ldots, v_n \right)^{\mathrm{T}}, \quad \nabla_x g^\varepsilon = \left(\nabla_x g_0^\varepsilon, \nabla_x g_1^\varepsilon, \ldots, \nabla_x g_n^\varepsilon \right).
$$

For each i which takes integer value from 0 to n, the v_i in velocity matrix v is the ith velocity defined in $\mathcal{V}$ and the ith component of the DVBGK Boltzmann equation (5) reads as

$$
\varepsilon \partial_t g_i^\varepsilon + v_i \cdot \nabla_x g_i^\varepsilon = \frac{1}{\varepsilon \nu} \left(g_{i,\mathrm{eq}}^\varepsilon - g_i^\varepsilon \right), \quad g_i^\varepsilon \big|_{t=0} = g_i^{0,\varepsilon},
$$

where

$$
g_{i,\mathrm{eq}}^\varepsilon := \rho^\varepsilon + \frac{v_i \cdot u^\varepsilon}{c_{\mathrm{s}}^2} + \frac{\varepsilon}{2 c_{\mathrm{s}}^4} \sum_{\alpha,\beta=1}^{d} u_\alpha^\varepsilon u_\beta^\varepsilon \left(v_{i,\alpha} v_{i,\beta} - c_{\mathrm{s}}^2 \delta_{\alpha\beta} \right)
\tag{6}
$$

with

$$
\rho^\varepsilon := \sum_{i=0}^{n} w_i g_i^\varepsilon, \quad u^\varepsilon := \sum_{i=0}^{n} w_i v_i g_i^\varepsilon.
\tag{7}
$$

It should be emphasised that the DVBGK Boltzmann equation (5), which is only discretised in particle velocity, is distinct from the lattice BGK Boltzmann equation which is fully discretised in space, time and particle velocity. Following the standard terminology, we call the numerical implementation of the DVBGK Boltzmann equation (5) on an isotropic lattice to be the D*d*Q*n* scheme, see e.g. [16]. As an important fact, we emphasise that the summation condition (4) implies that

$$\rho^{\varepsilon} = \sum_{i=0}^{n} w_i g_{i,\mathrm{eq}}^{\varepsilon}, \quad u^{\varepsilon} = \sum_{i=0}^{n} w_i v_i g_{i,\mathrm{eq}}^{\varepsilon}. \tag{8}$$

In order to achieve our goal, we follow the philosophy of Jiang *et al* [13] in using the energy method approach to solve the Boltzmann equation, and the fundamental idea of Bardos *et al* [4] in taking the hydrodynamic limits. Apart from the rigorous analysis which constructs a local solution to the incompressible Navier–Stokes equations from the DVBGK Boltzmann equation, our work also contains a numerical part, namely, we conduct computations in 2D to provide information about the rate with which the hydrodynamic limit is achieved when the Knudsen number tends to zero. More specifically, we work with the D2Q9 scheme to solve the DVBGK Boltzmann equation (5) by considering 9 weights

$$w_i = \begin{cases} \dfrac{4}{9} & \text{for} \quad i = 0, \\[2mm] \dfrac{1}{9} & \text{for} \quad i = 1, 2, 3, 4, \\[2mm] \dfrac{1}{36} & \text{for} \quad i = 5, 6, 7, 8 \end{cases} \tag{9}$$

and 9 velocities

$$\begin{cases} v_0 = (0,0), \\ v_1 = (1,0), & v_2 = (0,1), & v_3 = (-1,0), & v_4 = (0,-1), \\ v_5 = (1,1), & v_6 = (-1,1), & v_7 = (-1,-1), & v_8 = (1,-1) \end{cases} \tag{10}$$

with sound speed $c_{\mathrm{s}} = \frac{1}{\sqrt{3}}$.

1.1. Rigorous justification of the hydrodynamic limit: analysis part

In order to establish the solvability of the DVBGK Boltzmann equation in a simple way, we follow the philosophy in [11] to do cutoff to equation (5) in Fourier space, i.e. we work with the approximate equation of (5) instead of (5) itself. We shall see by the end of section 3 that this consideration would not affect the incompressible Navier–Stokes limit. To summarise the underlying reason in a philosophical sentence, the hydrodynamic limit is not sensitive to small changes on the form of the Boltzmann equation.

Suppose that $(\mathcal{V}, w)$ is an isotropic lattice. We next define the approximate equation of (5) on $(\mathcal{V}, w)$. Let $\varepsilon \in (0, 1)$. For $h \in L^2(\mathbf{R}^d)$, we define that

$$\Lambda_{\varepsilon}(h) := \int_{\mathbf{R}^d} 1_{|\xi| < \frac{1}{\varepsilon}}(\xi)\, \widehat{h}(\xi)\, \mathrm{e}^{2\pi i x \cdot \xi}\, \mathrm{d}\xi, \quad \widehat{h}(\xi) := \int_{\mathbf{R}^d} h(x)\, \mathrm{e}^{-2\pi i x \cdot \xi}\, \mathrm{d}x, \tag{11}$$

where $1_{|\xi| < \varepsilon^{-1}}$ represents the indicator function for the open ball $\{\xi \in \mathbf{R}^d \mid |\xi| < \varepsilon^{-1}\}$, i.e. $\Lambda_{\varepsilon}(h)$ is the cutoff in Fourier space for h with respect to spatial *x*-variable. Since the DVBGK

Boltzmann equation (5) is already discretised in velocity variable *v*, there is no need to do cutoff to it in Fourier space with respect to *v*. Hence, we define the approximate DVBGK Boltzmann equation to be the vector equation

$$\varepsilon \partial_t \Lambda_\varepsilon \left(g^\varepsilon \right) + v \cdot \nabla_x \Lambda_\varepsilon \left(g^\varepsilon \right) = \frac{1}{\varepsilon \nu} \left(\mathcal{G}_{\mathrm{eq}}^\varepsilon - \Lambda_\varepsilon \left(g^\varepsilon \right) \right), \quad \Lambda_\varepsilon \left(g^\varepsilon \right) \big|_{t=0} = \Lambda_\varepsilon \left(g^{0,\varepsilon} \right) \tag{12}$$

where

$$\Lambda_\varepsilon \left(g^\varepsilon \right) = \left(\Lambda_\varepsilon \left(g_0^\varepsilon \right), \ldots, \Lambda_\varepsilon \left(g_n^\varepsilon \right) \right), \quad \mathcal{G}_{\mathrm{eq}}^\varepsilon := \left(\mathcal{G}_{0,\mathrm{eq}}^\varepsilon, \ldots, \mathcal{G}_{n,\mathrm{eq}}^\varepsilon \right),$$
$$\Lambda_\varepsilon \left(g^{0,\varepsilon} \right) := \left(\Lambda_\varepsilon \left(g_0^{0,\varepsilon} \right), \ldots, \Lambda_\varepsilon \left(g_n^{0,\varepsilon} \right) \right).$$

For each *i* which takes integer value from 0 to *n*, the *i*th component of the approximate DVBGK Boltzmann equation (12) reads as

$$\varepsilon \partial_t \Lambda_\varepsilon \left(g_i^\varepsilon \right) + v_i \cdot \nabla_x \Lambda_\varepsilon \left(g_i^\varepsilon \right) = \frac{1}{\varepsilon \nu} \left(\mathcal{G}_{i,\mathrm{eq}}^\varepsilon - \Lambda_\varepsilon \left(g_i^\varepsilon \right) \right), \quad \Lambda_\varepsilon \left(g_i^\varepsilon \right) \big|_{t=0} = \Lambda_\varepsilon \left(g_i^{0,\varepsilon} \right)$$

where

$$\mathcal{G}_{i,\mathrm{eq}}^\varepsilon := \Lambda_\varepsilon \left(\rho^\varepsilon \right) + \frac{v_i \cdot \Lambda_\varepsilon \left(u^\varepsilon \right)}{c_{\mathrm{s}}^2} + \frac{\varepsilon}{2 c_{\mathrm{s}}^4} \sum_{\alpha, \beta = 1}^d \left(v_{i,\alpha} v_{i,\beta} - c_{\mathrm{s}}^2 \delta_{\alpha\beta} \right) \Lambda_\varepsilon \left(\Lambda_\varepsilon \left(u^\varepsilon \right)_\alpha \Lambda_\varepsilon \left(u^\varepsilon \right)_\beta \right),$$

$$\Lambda_\varepsilon \left(\rho^\varepsilon \right) := \sum_{i=0}^n w_i \Lambda_\varepsilon \left(g_i^\varepsilon \right), \quad \Lambda_\varepsilon \left(u^\varepsilon \right) := \sum_{i=0}^n w_i v_i \Lambda_\varepsilon \left(g_i^\varepsilon \right) \tag{13}$$

and $\Lambda_\varepsilon (u^\varepsilon)_\alpha$ denotes the αth component of $\Lambda_\varepsilon (u^\varepsilon)$. Analogous to (8), we deduce by (4) that

$$\Lambda_\varepsilon \left(\rho^\varepsilon \right) = \sum_{i=0}^n w_i \mathcal{G}_{i,\mathrm{eq}}^\varepsilon, \quad \Lambda_\varepsilon \left(u^\varepsilon \right) = \sum_{i=0}^n w_i v_i \mathcal{G}_{i,\mathrm{eq}}^\varepsilon. \tag{14}$$

Before stating our main results, we would like to define several notations and concepts that will repeatedly appear in this paper. For $f = \left(f(v_0), \ldots, f(v_n) \right)$, $h = \left(h(v_0), \ldots, h(v_n) \right) \in \mathbf{R}^{n+1}$, we define inner products

$$\langle f, h \rangle_{L_v^2} := \sum_{i=0}^n f(v_i) h(v_i), \quad \langle f, h \rangle_{L_{v,w}^2} := \sum_{i=0}^n w_i f(v_i) h(v_i).$$

For $f = \left(f(x, v_0), \ldots, f(x, v_n) \right)$, $h = \left(h(x, v_0), \ldots, h(x, v_n) \right) \in L^2(\mathbf{R}_x^d)$, we define inner products

$$\langle f, h \rangle_{L_x^2 L_v^2} := \sum_{i=0}^n \int_{\mathbf{R}^d} f(x, v_i) h(x, v_i) \, \mathrm{d}x, \quad \langle f, h \rangle_{L_x^2 L_{v,w}^2} := \sum_{i=0}^n w_i \int_{\mathbf{R}^d} f(x, v_i) h(x, v_i) \, \mathrm{d}x.$$

Let $\mathbf{N}_0 := \mathbf{N} \cup \{0\}$ and $\tau = (\tau_1, \tau_2, \ldots, \tau_d) \in \mathbf{N}_0^d$. We define

$$|\tau|_{\mathrm{s}} := \sum_{j=1}^d \tau_j$$

and use the notation ∂_x^τ to denote the differentiation $\partial_{x_1}^{\tau_1}\partial_{x_2}^{\tau_2}\ldots\partial_{x_d}^{\tau_d}$. For $f = \left(f(x,v_0),\ldots,f(x,v_n)\right) \in H^\eta(\mathbf{R}_x^d)$ with $\eta \in \mathbf{R}$, we define the norm

$$\|f\|_{H_x^\eta L_{v,w}^2}^2 := \sum_{i=0}^n w_i \|f(\cdot,v_i)\|_{H^\eta(\mathbf{R}_x^d)}^2.$$

We set $H_x^\eta L_{v,w}^2$ to be the Sobolev space $H^\eta(\mathbf{R}_x^d)$ equipped with norm $\|\cdot\|_{H_x^\eta L_{v,w}^2}$. By the Cauchy–Schwarz inequality, we can easily observe that the space $H_x^\eta L_{v,w}^2$ is equivalent to the standard Sobolev space $H^\eta(\mathbf{R}_x^d)$. Regarding the approximate equation (12), we establish its local well-posedness.

Lemma 2. *Suppose that $(\mathcal{V},w)$ is an isotropic lattice. Let $\varepsilon > 0$ be fixed and $m \in \mathbf{N}$ with $m > d$. Then, for any $g^{0,\varepsilon} \in H_x^m L_{v,w}^2$, there exist constants $T_0 = T_0(g^{0,\varepsilon}, \mathcal{V}, c_s, m, d) > 0$ and $c_0 = c_0(\mathcal{V}, c_s, m, d) > 0$, such that the approximate DVBGK Boltzmann equation (12) on $(\mathcal{V},w)$ with initial data $\Lambda_\varepsilon(g^{0,\varepsilon})$ is locally well-posed. In particular, the unique local solution*

$$g^\varepsilon \in L^\infty\left([0,T_0]; H_x^m L_{v,w}^2\right)$$

satisfies $g^\varepsilon\big|_{t=0} = \Lambda_\varepsilon(g^{0,\varepsilon})$, $g^\varepsilon = \Lambda_\varepsilon(g^\varepsilon)$ and the local energy inequality

$$\begin{aligned}
&\sup_{t\in[0,T_0]} \|g^\varepsilon(t)\|_{H_x^m L_{v,w}^2}^2 + \frac{1}{\varepsilon^2}\int_0^{T_0} \|\mathcal{G}_{\mathrm{eq}}^\varepsilon(s) - g^\varepsilon(s)\|_{H_x^m L_{v,w}^2}^2 \, ds \\
&\leqslant \frac{(\nu+\nu^2)\|\Lambda_\varepsilon(g^{0,\varepsilon})\|_{H_x^m L_{v,w}^2}^2}{\nu - c_0 T_0 \|\Lambda_\varepsilon(g^{0,\varepsilon})\|_{H_x^m L_{v,w}^2}^2} - \nu\|\Lambda_\varepsilon(g^{0,\varepsilon})\|_{H_x^m L_{v,w}^2}^2.
\end{aligned} \tag{15}$$

The merit of working with the approximate equation (12) instead of the original DVBGK Boltzmann equation (5) is that the existence of a local solution to (12) can be constructed easily using the classical Picard's method. Since we are considering the case where $m > d$, the Sobolev space $H^m(\mathbf{R}_x^d)$ is indeed a Banach algebra. As a result, the local energy estimate (15) can be derived easily by the traditional energy method. To show the existence of a local solution, for each $0 \leqslant i \leqslant n$, we consider a sequence of functions $\{f_{i,j}^\varepsilon(t)\}_{j\in\mathbf{N}_0}$ that is constructed inductively by

$$f_{i,j+1}^\varepsilon(t) := f_{i,0}^\varepsilon - \frac{1}{\varepsilon}\int_0^t v_i \cdot \nabla_x f_{i,j}^\varepsilon(s) \, ds + \frac{1}{\varepsilon^2\nu}\int_0^t \mathcal{G}_{i,\mathrm{eq},j}^\varepsilon(s) - f_{i,j}^\varepsilon(s) \, ds$$

for $j \geqslant 0$ where $f_{i,0}^\varepsilon = \Lambda_\varepsilon(g_{i,0}^\varepsilon)$,

$$\mathcal{G}_{i,\mathrm{eq},j}^\varepsilon := \Lambda_\varepsilon(\rho_j^\varepsilon) + \frac{v_i \cdot \Lambda_\varepsilon(u_j^\varepsilon)}{c_s^2} + \frac{\varepsilon}{2c_s^4}\sum_{\alpha,\beta=1}^d \left(v_{i,\alpha}v_{i,\beta} - c_s^2\delta_{\alpha\beta}\right)\Lambda_\varepsilon\left(\Lambda_\varepsilon(u_{j,\alpha}^\varepsilon)\Lambda_\varepsilon(u_{j,\beta}^\varepsilon)\right),$$

$$\Lambda_\varepsilon(\rho_j^\varepsilon) := \sum_{i=0}^n w_i f_{i,j}^\varepsilon, \quad \Lambda_\varepsilon(u_j^\varepsilon) := \sum_{i=0}^n w_i v_i f_{i,j}^\varepsilon.$$

Since $f_{i,j}^\varepsilon$ has compact support in Fourier space, by the Bernstein-type lemma (see e.g. [3, lemma 2.1]), we can control the H_x^m norm of $v_i \cdot \nabla_x f_{i,j}^\varepsilon$ by the H_x^m norm of $f_{i,j}^\varepsilon$. Although in this case ε with negative power would appear in the coefficient, this is harmless. Thus, we can construct a local solution using the simple Picard's iteration. Let $f_j^\varepsilon := (f_{i,j}^\varepsilon)_{0 \leqslant i \leqslant n}$. We can prove

by induction that the sequence $\{f_j^\varepsilon(t)\}_{j\in\mathbf{N}_0}$ is Cauchy in $L^\infty([0,T_*^\varepsilon];H_x^m L_{v,w}^2)$ with some $T_*^\varepsilon > 0$ that depends on ε. The existence of a local solution can then be concluded by the contraction mapping theorem and the Banach fixed point theorem. Since the existence time T_0 in the local energy inequality (15) is independent of ε, we can extend the local solution up to time T_0. Moreover, by analogous energy method, the uniqueness for the local solution and continuity with respect to initial data can be argued by deriving the energy inequality for the difference between two local solutions that are well-defined on the same time interval. This completes the proof of lemma 2.

Having the local solvability for the approximate equation (12), we can then follow the idea of Bardos *et al* [4] to take the hydrodynamic limit. The main convergence theorem of this paper reads as follows.

Theorem 3. *Let $m\in\mathbf{N}$ with $m>d$. For any $(\rho_0,u_0)\in H^m(\mathbf{R}_x^d)$ and $0<\varepsilon<1$, we consider the initial data $g^{0,\varepsilon} = (g_0^{0,\varepsilon},g_1^{0,\varepsilon},\ldots,g_n^{0,\varepsilon})$ with*

$$g_i^{0,\varepsilon} = \Lambda_\varepsilon(\rho_0) + \frac{v_i}{c_s^2}\cdot\Lambda_\varepsilon(u_0), \quad \forall\, 0\leqslant i \leqslant n \tag{16}$$

to the approximate DVBGK Boltzmann equation (12) where Λ_ε is the cutoff in Fourier space operator defined by (11). Suppose that $\{\varepsilon_n\}_{n\in\mathbf{N}}\subset(0,1)$ is a sequence which converges to 0 as $n\to\infty$. For each $n\in\mathbf{N}$, let g^{ε_n} be the unique local solution to the approximate DVBGK Boltzmann equation (12) established in lemma 2 corresponding to the initial data g^{0,ε_n}. Then, there exist a subsequence $\{\varepsilon_{n(k)}\}_{k\in\mathbf{N}}$ and $(\rho,u)\in L^\infty([0,T_0];H^m(\mathbf{R}_x^d))$ such that

$$g_i^{\varepsilon_{n(k)}} \overset{*}{\rightharpoonup} \rho + \frac{v_i\cdot u}{c_s^2} \quad \text{as}\quad k\to\infty$$

in the weak-$*$ topology $\sigma\big(L^\infty([0,T_0];H^m(\mathbf{R}_x^d));L^1([0,T_0];H^{-m}(\mathbf{R}_x^d))\big)$. Moreover, u is a local weak solution to the Cauchy problem of the incompressible Navier–Stokes equations

$$\begin{cases} \partial_t u - c_s^2\nu\Delta_x u + \nabla_x\cdot(u\otimes u) + \nabla_x p = 0, \\ \qquad\qquad\qquad\qquad\quad \nabla_x\cdot u = 0, \\ \qquad\qquad\qquad\qquad\quad u(x,0) = \mathbb{P}(u_0) \end{cases} \tag{17}$$

where $\mathbb{P}$ denotes the Helmholtz projection for $L^2(\mathbf{R}_x^d)$. Furthermore, $u\in C([0,T_0];H^{m-1}(\mathbf{R}_x^d))$ satisfies the local energy inequality

$$\|u\|_{L^\infty([0,T_0];H^m(\mathbf{R}_x^d))} \lesssim \|\rho_0\|_{H^m(\mathbf{R}_x^d)} + \|u_0\|_{H^m(\mathbf{R}_x^d)}.$$

The formal derivation of the incompressible Navier–Stokes equations is somehow standard. By taking the inner product of the approximate equation (12) with 1 and v in the $L_{v,w}^2$ sense, we obtain that

$$\begin{cases} \partial_t\rho^\varepsilon + \dfrac{1}{\varepsilon}\nabla_x\cdot u^\varepsilon = 0, \\ \partial_t u^\varepsilon + \displaystyle\sum_{i=0}^n \dfrac{1}{\varepsilon}w_i\nabla_x\cdot(v_i\otimes v_i g_i^\varepsilon) = 0. \end{cases} \tag{18}$$

It can be easily observed that the divergence free condition comes from the first equation of (18) in hydrodynamic limit. To derive the main part of the Navier–Stokes equations, which results

from the second equation of (18), more effort is needed. For each $0 \leqslant i \leqslant n$, we have to use the matrix $A_i := (v_i \otimes v_i) - c_{\mathrm{s}}^2 I$ to rewrite

$$\nabla_x \cdot (v_i \otimes v_i g_i^\varepsilon) = \nabla_x \cdot \left(A_i \left(g_i^\varepsilon - \mathcal{G}_{i,\mathrm{eq}}^\varepsilon\right)\right) + \nabla_x \cdot \left(A_i \mathcal{G}_{i,\mathrm{eq}}^\varepsilon\right) + c_{\mathrm{s}}^2 \nabla_x g_i^\varepsilon.$$

Then, by making use of the isotropic summation condition (4), we can show that the transport term $u^\varepsilon \cdot \nabla_x u^\varepsilon$ comes from the summation of $\varepsilon^{-1} w_i \nabla_x \cdot (A_i \mathcal{G}_{i,\mathrm{eq}}^\varepsilon)$ in i whereas the diffusion term $\Delta_x u^\varepsilon$ comes from the summation of $\varepsilon^{-1} w_i \nabla_x \cdot \left(A_i (g_i^\varepsilon - \mathcal{G}_{i,\mathrm{eq}}^\varepsilon)\right)$ in i. Hence, the second equation of (18) can be rewritten in the form of

$$\partial_t u^\varepsilon - c_{\mathrm{s}}^2 \nu \Delta_x u^\varepsilon + \nabla_x \cdot (u^\varepsilon \otimes u^\varepsilon) - c_{\mathrm{s}}^2 \nu \nabla_x \operatorname{div} u^\varepsilon + \frac{c_{\mathrm{s}}^2}{\varepsilon} \nabla_x \rho^\varepsilon + \mathcal{R}^\varepsilon = 0 \qquad (19)$$

where $\mathcal{R}^\varepsilon$ is a remainder term which converges to zero in the sense of distributions in the hydrodynamic limit. The term $\nabla_x \rho^\varepsilon$ in (19) is a bad term with high frequency, its coefficient is of order ε^{-1} which blows up as $\varepsilon \to 0$. We have to get rid of this effect from (19) before we take the limit $\varepsilon \to 0$. In order to do so, we apply the L^2 Helmholtz projection $\mathbb{P}$ to both sides of (19). The gradient term $\varepsilon^{-1} \nabla_x \rho^\varepsilon$ is thus eliminated. Without terms having high frequency, we can then show that $\mathbb{P}(u^\varepsilon)$ is equi-continuous in time t by an energy argument. Suppressing subsequences, the strong convergence of $\mathbb{P}(u^\varepsilon)$ to u as $\varepsilon \to 0$ can be concluded by the Arzelà–Ascoli theorem. Finally, the convergence of $u^\varepsilon - \mathbb{P}(u^\varepsilon)$ to zero in the sense of distributions is guaranteed by a compensated compactness result by Lions and Masmoudi [21]. It is worth noting that the strong convergence of $\mathbb{P}(u^\varepsilon)$ (lemma 15) guarantees that the initial data for the macroscopic fluid velocity is only determined by the first moment of $g^{0,\varepsilon}$, i.e. $\sum_{i=0}^n w_i v_i g_i^{0,\varepsilon}$. Hence, when we construct the microscopic initial data for the DVBGK Boltzmann system from the macroscopic fluid initial data, cf formula (16), it is not necessary to take into account the nonlinear effect in $g_{i,\mathrm{eq}}^\varepsilon$ since the matrix $v_i \otimes v_i - c_{\mathrm{s}}^2 I$ is orthogonal to 1 and v_i in the $L_{v,w}^2$ sense.

As the end of the analysis part, we give characterisations to isotropic lattices in the case when $d = 2, 3$ and $c_{\mathrm{s}} = 3^{-\frac{1}{2}}$. Since the combination of an isotropic lattice and speed of sound is scale invariant (see remark 26), we have to restrict the size of every component of each particle velocity in order to give a characterisation. The key idea for characterising 2D and 3D isotropic lattices is as follows. If we require the size condition that every component of each particle velocity takes value in the interval $[-1, 1]$, then every component of each particle velocity can only take value in the set $\{-1, 0, 1\}$. Moreover, we note that for $1 \leqslant i \leqslant 5$, the ith summation condition in (4) is indeed the expectations of all possible products of $i - 1$ components of particle velocities. The characterisations for 2D and 3D isotropic lattices (when $c_{\mathrm{s}} = 3^{-\frac{1}{2}}$) can be obtained by algebraic manipulations of summation conditions in (4) in the language of expectations. In particular, in the 2D case with $c_{\mathrm{s}} = 3^{-\frac{1}{2}}$, we show that if we require that each component of every particle velocity to take value in $[-1, 1]$, then the D2Q9 scheme is the only possible isotropic lattice. In the 3D case with $c_{\mathrm{s}} = 3^{-\frac{1}{2}}$, we have more than one possible choice for isotropic lattices, simply because of the system is underdetermined.

1.2. Numerical investigation of the hydrodynamic limit

In order to illustrate these concepts, in particular, the hydrodynamic limit of the DVBGK Boltzmann system, we conduct numerical simulations of system (5) for different values of ε and compare these solutions to the solution of the Navier–Stokes system (17) with the corresponding macroscopic initial condition. It should be emphasised here that, since we are interested in the question of the convergence of solutions to one PDE problem, i.e. (5), to solutions

of another PDE problem, (17), as a parameter in the former is varied, we choose to rely on the most accurate numerical approach available, which is the pseudospectral method [7], rather than the LBM. Moreover, in most cases for the LBM, the Knudsen number does not appear as an explicit parameter in (1), and its smallness can only be implicitly guaranteed by properly adjusting the ratio between the spatial discretisation Δx and the temporal discretisation Δt. On the other hand, our numerical approach is based directly on system (5) where the Knudsen number is explicitly present such that that a direct quantitative comparison with the solutions of (17) is possible.

For simplicity, we focus on the 2D case ($d = 2$) and consider problems defined on a periodic spatial domain (i.e. a 2D torus $\mathbf{T}_x^2$). Flows corresponding to two different macroscopic initial conditions are studied, namely, the Taylor–Green vortex for which the 2D Navier–Stokes system admits a closed-form analytic solution and its perturbed version which leads to a turbulent-like evolution featuring repeated filamentation of vortices resulting in an enstrophy cascade. In both cases we observe that $\|\nabla_x u(T) - \nabla_x u^\varepsilon(T)\|_{L^2(\mathbf{T}_x^2)} = \mathcal{O}(\varepsilon^2)$ at some time $T > 0$ as $\varepsilon \to 0$. This observation agrees with the convergence rate proved by Junk and Yong [15] using the Hilbert expansion. For the BGL program, it is very hard to establish this convergence rate rigorously, cf section 7 for a discussions of the underlying reasons.

1.3. Organisation of the paper

This article is organised as follows. Section 2 is devoted to the local well-posedness of the approximate DVBGK Boltzmann equation (12). In section 2.1, we employ the standard energy method to derive a local energy inequality which holds for both the approximate and the original DVBGK Boltzmann equation. In section 2.2, we construct a local solution to the approximate equation (12) using the classical Picard's method. In section 2.3, we consider analogous energy argument as in section 2.1 to show the uniqueness and continuity with respect to initial data. Section 3 is devoted to the formal derivation of the incompressible Navier–Stokes equations. In particular, the transport term $u^\varepsilon \cdot \nabla_x u^\varepsilon$ is derived in section 3.1 and the diffusion term is derived in section 3.2. In section 4, we take the hydrodynamic limit and prove the convergence of the unique local solution for the approximate equation (12) to a local weak solution for the incompressible Navier–Stokes equations (17). In section 4.1, we apply the Helmholtz decomposition $\mathbb{P}$ to get rid of the gradient term that has high frequency in the formal equations derived in section 3. We also prove the strong convergence $\mathbb{P}(u^\varepsilon)$ to u here. In section 4.2, we collect all the convergence results established to give a proof to theorem 3. In section 5, we give characterisations to 2D and 3D isotropic lattices when $c_s = 3^{-\frac{1}{2}}$ and a size condition is imposed on particle velocities. In section 5.1, we rewrite the summation conditions in (4) using expectations in the probability setting. Section 5.2 is devoted to the characterisations of 2D and 3D isotropic lattices. In section 6, we present the numerical implementation of the D2Q9 scheme in the 2D case to provide numerical evidence that justify the convergence behaviour we have proved in the analysis part.

Throughout this paper, the notation $A \lesssim B$ will mean that there exists a constant c, which is independent of ε and ν, such that $A \leqslant cB$.

2. Local solvability of the approximate DVBGK Boltzmann equation

First of all, we would like to highlight a simple tool that is crucial for norm estimations of nonlinear terms in this paper.

Proposition 4. *For $m \in \mathbf{N}$ such that $m > d$, the Sobolev space $H^m(\mathbf{R}_x^d)$ is a Banach algebra.*

Proof. For $f, g \in H^m(\mathbf{R}_x^d)$, we can easily observe by the Cauchy–Schwarz inequality that

$$\|fg\|_{H^m(\mathbf{R}_x^d)}^2 \leqslant \sum_{\tau \in \mathbf{N}_0^d, |\tau|_s \leqslant m} (|\tau|_s + 1)^d \sum_{\sigma \in \mathbf{N}_0^d, \sigma \leqslant \tau} \|\partial_x^\sigma f \partial_x^{\tau - \sigma} g\|_{L^2(\mathbf{R}_x^d)}^2$$

where the notation $\sigma \leqslant \tau$ means that $\sigma_i \leqslant \tau_i$ for all $1 \leqslant i \leqslant d$. Since $m \geqslant d + 1$, we note that either $|\sigma|_s$ or $|\tau - \sigma|_s = |\tau|_s - |\sigma|_s$ must be less than or equal to $m - [\frac{d}{2}] - 1$ where $[\frac{d}{2}]$ denotes the largest integer less than or equal to $\frac{d}{2}$. Without loss of generality, we may assume that $|\sigma|_s \leqslant m - [\frac{d}{2}] - 1$. Then, by the continuous Sobolev embedding $H^{[\frac{d}{2}]+1}(\mathbf{R}_x^d) \hookrightarrow L^\infty(\mathbf{R}_x^d)$, we deduce that

$$\|\partial_x^\sigma f \partial_x^{\tau - \sigma} g\|_{L^2(\mathbf{R}_x^d)}^2 \leqslant \|\partial_x^\sigma f\|_{L^\infty(\mathbf{R}_x^d)}^2 \|\partial_x^{\tau - \sigma} g\|_{L^2(\mathbf{R}_x^d)}^2$$
$$\leqslant \|f\|_{H^m(\mathbf{R}_x^d)}^2 \|\partial_x^{\tau - \sigma} g\|_{L^2(\mathbf{R}_x^d)}^2 .$$

Hence,

$$\|fg\|_{H^m(\mathbf{R}_x^d)}^2 \leqslant \|f\|_{H^m(\mathbf{R}_x^d)}^2 \sum_{\tau \in \mathbf{N}_0^d, |\tau|_s \leqslant m} (|\tau|_s + 1)^d \|g\|_{H^{|\tau|_s}(\mathbf{R}_x^d)}^2$$
$$\leqslant (m+1)^d \|f\|_{H^m(\mathbf{R}_x^d)}^2 \|g\|_{H^m(\mathbf{R}_x^d)}^2 \cdot \sum_{j=0}^m \binom{j + d - 1}{d - 1}$$
$$\leqslant (m+1)^{d+1} \frac{(m+d-1)^{d-1}}{(d-1)!} \|f\|_{H^m(\mathbf{R}_x^d)}^2 \|g\|_{H^m(\mathbf{R}_x^d)}^2 .$$

This completes the proof of proposition 4 since $H^m(\mathbf{R}_x^d)$ is certainly a Banach space. $\qquad\square$

Remark 5. The continuous Sobolev embedding $H^{[\frac{d}{2}]+1}(\mathbf{R}_x^d) \hookrightarrow L^\infty(\mathbf{R}_x^d)$ plays an important role in proving proposition 4. We recall that in fact, it holds more generally that $H^s(\mathbf{R}_x^d) \hookrightarrow L^\infty(\mathbf{R}_x^d)$ whenever $s > \frac{d}{2}$ as

$$\|h\|_{L^\infty(\mathbf{R}_x^d)} \leqslant \|\widehat{h}\|_{L^1(\mathbf{R}_x^d)} \leqslant \|\langle \xi \rangle^{-s}\|_{L^2(\mathbf{R}_x^d)} \|\langle \xi \rangle^s \widehat{h}\|_{L^2(\mathbf{R}_x^d)} \lesssim \|h\|_{H^s(\mathbf{R}_x^d)}$$

where $\langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}}$ and $\widehat{h}$ denotes the Fourier transform of h.

2.1. Local energy estimate for the DVBGK Boltzmann equation

The local energy inequality can be derived by working directly with the original DVBGK Boltzmann equation (5).

Lemma 6. *Suppose that* $(\mathcal{V}, w)$ *is an isotropic lattice. Let* $\varepsilon > 0$ *and* $m \in \mathbf{N}$ *satisfying* $m > d$. *Then, for any* $g^{0,\varepsilon} \in H_x^m L_{v,w}^2$, *there exists* $T_0 = T_0(g^{0,\varepsilon}, \mathcal{V}, c_s, m, d) > 0$ *such that for any* $g^\varepsilon \in L^\infty([0, T_0]; H_x^m L_{v,w}^2)$ *satisfying the DVBGK Boltzmann equation* (5) *on* $(\mathcal{V}, w)$, *the local energy estimate*

$$\|g^\varepsilon(t)\|^2_{H^m_x L^2_{v,w}} + \frac{1}{\varepsilon^2} \int_0^t \|g^\varepsilon_{\mathrm{eq}}(s) - g^\varepsilon(s)\|^2_{H^m_x L^2_{v,w}} \, \mathrm{d}s$$
$$\leqslant \frac{(\nu + \nu^2) \|g^{0,\varepsilon}\|^2_{H^m_x L^2_{v,w}}}{\nu - t \cdot C_{LE} \|g^{0,\varepsilon}\|^2_{H^m_x L^2_{v,w}}} - \nu \|g^{0,\varepsilon}\|^2_{H^m_x L^2_{v,w}} \tag{20}$$

holds for any $t \in [0, T_0]$ with a constant $C_{LE} = C_{LE}(\mathcal{V}, c_{\mathrm{s}}, m, d) > 0$.

Proof. Let $\tau \in \mathbf{N}_0^d$ satisfying $|\tau|_{\mathrm{s}} \leqslant m$. By applying ∂_x^τ to the DVBGK Boltzmann equation (5) and taking its inner product with $\partial_x^\tau g^\varepsilon$ in the $L^2_x L^2_{v,w}$ sense, we obtain that

$$\frac{1}{2} \sum_{i=0}^n w_i \frac{\mathrm{d}}{\mathrm{d}t} \|\partial_x^\tau g_i^\varepsilon\|^2_{L^2(\mathbf{R}_x^d)} + \frac{1}{\varepsilon^2 \nu} \sum_{i=0}^n w_i \|\partial_x^\tau g_{i,\mathrm{eq}}^\varepsilon - \partial_x^\tau g_i^\varepsilon\|^2_{L^2(\mathbf{R}_x^d)}$$
$$= \frac{1}{\varepsilon^2 \nu} \sum_{i=0}^n w_i \int_{\mathbb{R}^d} \left(\partial_x^\tau g_{i,\mathrm{eq}}^\varepsilon - \partial_x^\tau g_i^\varepsilon \right) \partial_x^\tau g_{i,\mathrm{eq}}^\varepsilon \, \mathrm{d}x. \tag{21}$$

By (7) and (8), we observe that

$$\sum_{i=0}^n w_i \left(g_{i,\mathrm{eq}}^\varepsilon - g_i^\varepsilon \right) = 0, \quad \sum_{i=0}^n w_i v_i \left(g_{i,\mathrm{eq}}^\varepsilon - g_i^\varepsilon \right) = 0.$$

Hence, by substituting expression (6) for $g_{i,\mathrm{eq}}^\varepsilon$ into the right hand side of (21), we deduce that

$$\frac{1}{\varepsilon^2 \nu} \sum_{i=0}^n w_i \int_{\mathbb{R}^d} \left(\partial_x^\tau g_{i,\mathrm{eq}}^\varepsilon - \partial_x^\tau g_i^\varepsilon \right) \partial_x^\tau g_{i,\mathrm{eq}}^\varepsilon \, \mathrm{d}x$$
$$= \frac{1}{2c_{\mathrm{s}}^4 \varepsilon \nu} \sum_{i=0}^n \sum_{\alpha,\beta=1}^d \sum_{\sigma \in \mathbf{N}_0^d, \, \sigma \leqslant \tau} w_i \left(v_{i,\alpha} v_{i,\beta} - c_{\mathrm{s}}^2 \delta_{\alpha\beta} \right) \int_{\mathbb{R}^d} \left(\partial_x^\tau g_{i,\mathrm{eq}}^\varepsilon - \partial_x^\tau g_i^\varepsilon \right) \partial_x^\sigma u_\alpha^\varepsilon \partial_x^{\tau-\sigma} u_\beta^\varepsilon \, \mathrm{d}x$$
$$\leqslant \frac{\ell_{\mathcal{V},c_{\mathrm{s}}}}{2c_{\mathrm{s}}^4 \varepsilon \nu} \sum_{i=0}^n w_i \|\partial_x^\tau g_{i,\mathrm{eq}}^\varepsilon - \partial_x^\tau g_i^\varepsilon\|_{L^2(\mathbf{R}_x^d)} \sum_{\alpha,\beta=1}^d \sum_{\sigma \in \mathbf{N}_0^d, \, \sigma \leqslant \tau} \|\partial_x^\sigma u_\alpha^\varepsilon \partial_x^{\tau-\sigma} u_\beta^\varepsilon\|_{L^2(\mathbf{R}_x^d)}$$
$$\leqslant \frac{1}{2\varepsilon^2 \nu} \sum_{i=0}^n w_i \|\partial_x^\tau g_{i,\mathrm{eq}}^\varepsilon - \partial_x^\tau g_i^\varepsilon\|^2_{L^2(\mathbf{R}_x^d)} + \frac{\ell_{\mathcal{V},c_{\mathrm{s}}}^2 d^2 |\tau|_{\mathrm{s}}^d}{8c_{\mathrm{s}}^8 \nu} \sum_{\alpha,\beta=1}^d \sum_{\sigma \in \mathbf{N}_0^d, \, \sigma \leqslant \tau} \|\partial_x^\sigma u_\alpha^\varepsilon \partial_x^{\tau-\sigma} u_\beta^\varepsilon\|^2_{L^2(\mathbf{R}_x^d)}, \tag{22}$$

where $\ell_{\mathcal{V},c_{\mathrm{s}}} := \left(\max_{1 \leqslant i \leqslant n} |v_i| \right)^2 + c_{\mathrm{s}}^2$. From the proof of proposition 4, we see that

$$\sum_{\sigma \in \mathbf{N}_0^d, \, \sigma \leqslant \tau} \|\partial_x^\sigma u_\alpha^\varepsilon \partial_x^{\tau-\sigma} u_\beta^\varepsilon\|^2_{L^2(\mathbf{R}_x^d)} \leqslant \|u_\alpha^\varepsilon\|^2_{H^m(\mathbf{R}_x^d)} \|u_\beta^\varepsilon\|^2_{H^{|\tau|_{\mathrm{s}}}(\mathbf{R}_x^d)}.$$

On the other hand, we observe by (7) that the estimate

$$\|u_\alpha^\varepsilon\|^2_{H^k(\mathbf{R}_x^d)} \leqslant \left(\sum_{i=0}^n w_i \|g_i^\varepsilon\|_{H^k(\mathbf{R}_x^d)} \right)^2 \leqslant \sum_{i=0}^n w_i \|g_i^\varepsilon\|^2_{H^k(\mathbf{R}_x^d)}$$

holds for any $1 \leqslant \alpha \leqslant d$ and $k \in \mathbf{N}_0$. Therefore, (21) and (22) imply that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=0}^{n}w_i\|\partial_x^\tau g_i^\varepsilon\|_{L^2(\mathbf{R}_x^d)}^2 + \frac{1}{2\varepsilon^2\nu}\sum_{i=0}^{n}w_i\|\partial_x^\tau g_{i,\mathrm{eq}}^\varepsilon - \partial_x^\tau g_i^\varepsilon\|_{L^2(\mathbf{R}_x^d)}^2 \\
&\qquad \leqslant \frac{\ell_{\mathcal{V},c_{\mathrm{s}}}^2 d^4 |\tau|_{\mathrm{s}}^d}{8c_{\mathrm{s}}^8\nu}\|g^\varepsilon\|_{H_x^m L_{v,w}^2}^2\|g^\varepsilon\|_{H_x^{|\tau|_{\mathrm{s}}} L_{v,w}^2}^2.
\end{aligned}
\tag{23}
$$

Summing up inequality (23) over all $\tau \in \mathbf{N}_0^d$ satisfying $|\tau|_{\mathrm{s}} \leqslant m$ gives us

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|g^\varepsilon\|_{H_x^m L_{v,w}^2}^2 + \frac{1}{2\varepsilon^2\nu}\|g_{\mathrm{eq}}^\varepsilon - g^\varepsilon\|_{H_x^m L_{v,w}^2}^2 \leqslant \frac{\ell_{\mathcal{V},c_{\mathrm{s}}}^2 d^4 m^d}{8c_{\mathrm{s}}^8\nu}\|g^\varepsilon\|_{H_x^m L_{v,w}^2}^4.
\tag{24}
$$

Applying Gronwall's inequality (see e.g. [24, page 362]) to

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|g^\varepsilon\|_{H_x^m L_{v,w}^2}^2 \leqslant \frac{\ell_{\mathcal{V},c_{\mathrm{s}}}^2 d^4 m^d}{4c_{\mathrm{s}}^8\nu}\|g^\varepsilon\|_{H_x^m L_{v,w}^2}^4,
$$

we deduce that

$$
\|g^\varepsilon(t)\|_{H_x^m L_{v,w}^2}^2 \leqslant \frac{\nu\|g^{0,\varepsilon}\|_{H_x^m L_{v,w}^2}^2}{\nu - t \cdot C_{LE}\|g^{0,\varepsilon}\|_{H_x^m L_{v,w}^2}^2},
\tag{25}
$$

where $C_{LE} = C_{LE}(\mathcal{V}, c_{\mathrm{s}}, m, d) := 4^{-1}c_{\mathrm{s}}^{-8}\ell_{\mathcal{V},c_{\mathrm{s}}}^2 d^4 m^d$. Substituting inequality (25) into the inequality

$$
\frac{1}{\varepsilon^2 C_{LE}}\|g_{\mathrm{eq}}^\varepsilon - g^\varepsilon\|_{H_x^m L_{v,w}^2}^2 \leqslant \|g^\varepsilon\|_{H_x^m L_{v,w}^2}^4,
$$

which is another implication of inequality (24), we can further deduce that

$$
\frac{1}{\varepsilon^2}\int_0^t \|g_{\mathrm{eq}}^\varepsilon - g^\varepsilon\|_{H_x^m L_{v,w}^2}^2 \, \mathrm{d}s \leqslant \frac{\nu^2\|g^{0,\varepsilon}\|_{H_x^m L_{v,w}^2}^2}{\nu - t \cdot C_{LE}\|g^{0,\varepsilon}\|_{H_x^m L_{v,w}^2}^2} - \nu\|g^{0,\varepsilon}\|_{H_x^m L_{v,w}^2}^2.
$$

Finally, we take $T_0 < \nu C_{LE}^{-1}\|g^{0,\varepsilon}\|_{H_x^m L_{v,w}^2}^{-2}$. This completes the proof of lemma 6. $\square$

Remark 7. It is easy to observe that the cutoff operator Λ_ε commutes with the differentiation ∂_x. Moreover, it holds that for any $f, h \in L_x^2 L_{v,w}^2$,

$$
\langle \Lambda_\varepsilon(f), h\rangle_{L_x^2 L_{v,w}^2} = \langle f, \Lambda_\varepsilon(h)\rangle_{L_x^2 L_{v,w}^2} \quad \text{and} \quad \Lambda_\varepsilon(\Lambda_\varepsilon(f)) = \Lambda_\varepsilon(f).
$$

Making use of these three properties of the cutoff operator Λ_ε, we can deduce that lemma 6 also holds for the approximate DVBGK Boltzmann equation (12) defined on an isotropic lattice $(\mathcal{V}, w)$, with $g^{0,\varepsilon}$, g^ε and $g_{\mathrm{eq}}^\varepsilon$ in inequality (20) being replaced respectively by $\Lambda_\varepsilon(g^{0,\varepsilon})$, $\Lambda_\varepsilon(g^\varepsilon)$ and $\mathcal{G}_{\mathrm{eq}}^\varepsilon$.

2.2. Existence of a local solution for the approximate equation

Proof of lemma 2 (Existence). For simplicity of notations, we denote $f^\varepsilon = (f_i^\varepsilon)_{0 \leqslant i \leqslant n} :=$ $\Lambda_\varepsilon(g^\varepsilon)$ and $f_0^\varepsilon = (f_{i,0}^\varepsilon)_{0 \leqslant i \leqslant n} := \Lambda_\varepsilon(g^{0,\varepsilon})$, i.e.

$$f_i^\varepsilon = \Lambda_\varepsilon(g_i^\varepsilon) \quad \text{and} \quad f_{i,0}^\varepsilon = \Lambda_\varepsilon\left(g_i^{0,\varepsilon}\right), \quad 0 \leqslant i \leqslant n.$$

Performing integration of the ith component of the approximate DVBGK Boltzmann equation (12) for each $0 \leqslant i \leqslant n$ with respect to the time variable, we have that

$$f_i^\varepsilon(t) = f_{i,0}^\varepsilon - \int_0^t \frac{1}{\varepsilon}\left(v_i \cdot \nabla_x f_i^\varepsilon\right)\mathrm{d}s + \int_0^t \frac{1}{\varepsilon^2\nu}\left(\mathcal{G}_{i,\mathrm{eq}}^\varepsilon - f_i^\varepsilon\right)\mathrm{d}s. \tag{26}$$

Based on the integral equation (26), for each $0 \leqslant i \leqslant n$, we define a sequence of functions $\{f_{i,j}^\varepsilon\}_{j \in \mathbf{N}_0}$ inductively by

$$f_{i,j+1}^\varepsilon(t) = f_{i,0}^\varepsilon - \int_0^t \frac{1}{\varepsilon}\left(v_i \cdot \nabla_x f_{i,j}^\varepsilon(s)\right)\mathrm{d}s + \int_0^t \frac{1}{\varepsilon^2\nu}\left(\mathcal{G}_{i,\mathrm{eq},j}^\varepsilon(s) - f_{i,j}^\varepsilon(s)\right)\mathrm{d}s \tag{27}$$

for $j \in \mathbf{N}_0$ where

$$\mathcal{G}_{i,\mathrm{eq},j}^\varepsilon := \Lambda_\varepsilon\left(\rho_j^\varepsilon\right) + \frac{v_i \cdot \Lambda_\varepsilon\left(u_j^\varepsilon\right)}{c_\mathrm{s}^2} + \frac{\varepsilon}{2c_\mathrm{s}^4}\sum_{\alpha,\beta=1}^d \left(v_{i,\alpha}v_{i,\beta} - c_\mathrm{s}^2\delta_{\alpha\beta}\right)\Lambda_\varepsilon\left(\Lambda_\varepsilon\left(u_j^\varepsilon\right)_\alpha\Lambda_\varepsilon\left(u_j^\varepsilon\right)_\beta\right),$$

$$\Lambda_\varepsilon\left(\rho_j^\varepsilon\right) := \sum_{i=0}^n w_i f_{i,j}^\varepsilon, \quad \Lambda_\varepsilon\left(u_j^\varepsilon\right) := \sum_{i=0}^n w_i v_i f_{i,j}^\varepsilon$$

with $\Lambda_\varepsilon(u_j^\varepsilon)_\alpha$ denoting the αth component of $\Lambda_\varepsilon(u_j^\varepsilon)$. For every $j \in \mathbf{N}$, we set $f_j^\varepsilon := (f_{i,j}^\varepsilon)_{0 \leqslant i \leqslant n}$. The key idea here is to show that the vector sequence $\{f_j^\varepsilon\}_{j \in \mathbf{N}_0}$ is Cauchy in $L^\infty([0,T]; H_x^m L_{v,w}^2)$ for each $0 \leqslant i \leqslant n$. By Minkowski's integral inequality, for any $j \in \mathbf{N}$ and $0 \leqslant i \leqslant n$, we have that

$$
\begin{aligned}
\|f_{i,j+1}^\varepsilon - f_{i,j}^\varepsilon\|_{H^m(\mathbf{R}_x^d)} &\lesssim \int_0^t \frac{1}{\varepsilon}\|v_i \cdot \nabla_x\left(f_{i,j}^\varepsilon - f_{i,j-1}^\varepsilon\right)\|_{H^m(\mathbf{R}_x^d)}\mathrm{d}s \\
&\quad + \int_0^t \frac{1}{\varepsilon^2\nu}\|\mathcal{G}_{i,\mathrm{eq},j}^\varepsilon - \mathcal{G}_{i,\mathrm{eq},j-1}^\varepsilon\|_{H^m(\mathbf{R}_x^d)}\mathrm{d}s \\
&\quad + \int_0^t \frac{1}{\varepsilon^2\nu}\|f_{i,j}^\varepsilon - f_{i,j-1}^\varepsilon\|_{H^m(\mathbf{R}_x^d)}\mathrm{d}s.
\end{aligned}
\tag{28}
$$

Since the Fourier transform of $f_{i,j}^\varepsilon$ is supported within $\overline{B_{\varepsilon^{-1}}(0)}$ for any $0 \leqslant i \leqslant n$ and $j \in \mathbf{N}_0$, by the Bernstein-type lemma (see [3, lemma 2.1]), we deduce that

$$\|v_i \cdot \nabla_x\left(f_{i,j}^\varepsilon - f_{i,j-1}^\varepsilon\right)\|_{H^m(\mathbf{R}_x^d)} \lesssim \|\nabla_x\left(f_{i,j}^\varepsilon - f_{i,j-1}^\varepsilon\right)\|_{H^m(\mathbf{R}_x^d)} \lesssim \frac{1}{\varepsilon}\|f_{i,j}^\varepsilon - f_{i,j-1}^\varepsilon\|_{H^m(\mathbf{R}_x^d)}. \tag{29}$$

To estimate the H^m norm of $\mathcal{G}_{i,\mathrm{eq},j}^\varepsilon - \mathcal{G}_{i,\mathrm{eq},j-1}^\varepsilon$, we first observe that for any $j \in \mathbf{N}$,

$$\|\Lambda_\varepsilon\left(\rho_j^\varepsilon\right) - \Lambda_\varepsilon\left(\rho_{j-1}^\varepsilon\right)\|_{H^m(\mathbf{R}_x^d)} + \|v_i \cdot \left(\Lambda_\varepsilon\left(u_j^\varepsilon\right) - \Lambda_\varepsilon\left(u_{j-1}^\varepsilon\right)\right)\|_{H^m(\mathbf{R}_x^d)} \lesssim \|f_j^\varepsilon - f_{j-1}^\varepsilon\|_{H_x^m L_{v,w}^2}. \tag{30}$$

Since the cutoff operator Λ_ε commutes with the differentiation ∂_x, the nonlinear terms in $\mathcal{G}^\varepsilon_{i,\mathrm{eq},j} - \mathcal{G}^\varepsilon_{i,\mathrm{eq},j-1}$ can be estimated by Plancherel's identity and proposition 4, i.e. for any $j \in \mathbf{N}$ and $1 \leqslant \alpha, \beta \leqslant d$, it holds that

$$
\begin{aligned}
&\left\| \Lambda_\varepsilon \left(\Lambda_\varepsilon \left(u^\varepsilon_j \right)_\alpha \Lambda_\varepsilon \left(u^\varepsilon_j \right)_\beta - \Lambda_\varepsilon \left(u^\varepsilon_{j-1} \right)_\alpha \Lambda_\varepsilon \left(u^\varepsilon_{j-1} \right)_\beta \right) \right\|_{H^m(\mathbf{R}^d_x)} \\
&\quad \leqslant \left\| \left(\Lambda_\varepsilon \left(u^\varepsilon_j \right)_\alpha - \Lambda_\varepsilon \left(u^\varepsilon_{j-1} \right)_\alpha \right) \Lambda_\varepsilon \left(u^\varepsilon_j \right)_\beta \right\|_{H^m(\mathbf{R}^d_x)} \\
&\qquad + \left\| \Lambda_\varepsilon \left(u^\varepsilon_{j-1} \right)_\alpha \left(\Lambda_\varepsilon \left(u^\varepsilon_j \right)_\beta - \Lambda_\varepsilon \left(u^\varepsilon_{j-1} \right)_\beta \right) \right\|_{H^m(\mathbf{R}^d_x)} \\
&\quad \lesssim \left(\left\| \Lambda_\varepsilon \left(u^\varepsilon_j \right) \right\|_{H^m(\mathbf{R}^d_x)} + \left\| \Lambda_\varepsilon \left(u^\varepsilon_{j-1} \right) \right\|_{H^m(\mathbf{R}^d_x)} \right) \left\| \Lambda_\varepsilon \left(u^\varepsilon_j \right) - \Lambda_\varepsilon \left(u^\varepsilon_{j-1} \right) \right\|_{H^m(\mathbf{R}^d_x)} \\
&\quad \lesssim \left(\| f^\varepsilon_j \|_{H^m_x L^2_{v,w}} + \| f^\varepsilon_{j-1} \|_{H^m_x L^2_{v,w}} \right) \| f^\varepsilon_j - f^\varepsilon_{j-1} \|_{H^m_x L^2_{v,w}}.
\end{aligned}
\tag{31}
$$

Hence, by substituting estimates (29)–(31) back into inequality (28), we obtain that for any $j \in \mathbf{N}$ and $0 \leqslant i \leqslant n$,

$$
\begin{aligned}
\| f^\varepsilon_{i,j+1} - f^\varepsilon_{i,j} \|_{H^m(\mathbf{R}^d_x)} &\lesssim \int_0^t \frac{1+\nu}{\varepsilon^2 \nu} \| f^\varepsilon_{i,j} - f^\varepsilon_{i,j-1} \|_{H^m(\mathbf{R}^d_x)} \, \mathrm{d}s \\
&\quad + \int_0^t \frac{1}{\varepsilon^2 \nu} \left(1 + \| f^\varepsilon_j \|_{H^m_x L^2_{v,w}} + \| f^\varepsilon_{j-1} \|_{H^m_x L^2_{v,w}} \right) \| f^\varepsilon_j - f^\varepsilon_{j-1} \|_{H^m_x L^2_{v,w}} \, \mathrm{d}s,
\end{aligned}
\tag{32}
$$

as $\varepsilon \in (0,1)$.

Finally, we shall prove by induction that there exists $T > 0$ sufficiently small so that simultaneously, it holds for any $j \in \mathbf{N}$ that

$$
\begin{aligned}
\| f^\varepsilon_j \|_{L^\infty_T H^m_x L^2_{v,w}} &:= \sup_{t \in [0,T]} \| f^\varepsilon_j(t) \|_{H^m_x L^2_{v,w}} \\
&\lesssim \| f^\varepsilon_0 \|_{H^m_x L^2_{v,w}} + \frac{2T(1+\nu)}{\varepsilon^2 \nu} \left(\| f^\varepsilon_0 \|_{H^m_x L^2_{v,w}} + \| f^\varepsilon_0 \|^2_{H^m_x L^2_{v,w}} \right) =: M(f^\varepsilon_0, T)
\end{aligned}
\tag{33}
$$

and for any $j \in \mathbf{N}$ satisfying $j \geqslant 2$,

$$
\| f^\varepsilon_j - f^\varepsilon_{j-1} \|_{L^\infty_T H^m_x L^2_{v,w}} \leqslant \frac{1}{2} \| f^\varepsilon_{j-1} - f^\varepsilon_{j-2} \|_{L^\infty_T H^m_x L^2_{v,w}}.
\tag{34}
$$

Let $k \in \mathbf{N}$ with $k \geqslant 2$. Suppose that estimates (33) and (34) hold simultaneously for all $1 \leqslant j \leqslant k$. Then, by summing up estimate (32) over $0 \leqslant i \leqslant n$, we deduce that

$$
\| f^\varepsilon_{k+1} - f^\varepsilon_k \|_{L^\infty_T H^m_x L^2_{v,w}} \lesssim \frac{(1+\nu)T}{\varepsilon^2 \nu} \| f^\varepsilon_k - f^\varepsilon_{k-1} \|_{L^\infty_T H^m_x L^2_{v,w}} \left(1 + \| f^\varepsilon_{k-1} \|_{L^\infty_T H^m_x L^2_{v,w}} + \| f^\varepsilon_k \|_{L^\infty_T H^m_x L^2_{v,w}} \right).
$$

Using assumption (33) for cases $j = k - 1$ and $j = k$, we have that

$$
\| f^\varepsilon_{k+1} - f^\varepsilon_k \|_{L^\infty_T H^m_x L^2_{v,w}} \lesssim \frac{(1+\nu)T}{\varepsilon^2 \nu} \left(1 + 2M(f^\varepsilon_0, T) \right) \| f^\varepsilon_k - f^\varepsilon_{k-1} \|_{L^\infty_T H^m_x L^2_{v,w}}.
$$

There exists $T_* = T_*(f^\varepsilon_0, \varepsilon, \nu) > 0$ such that

$$
\frac{(1+\nu)T_*}{\varepsilon^2 \nu} \left(1 + M(f^\varepsilon_0, T_*) \right) < \frac{1}{2}.
$$

Hence,

$$\|f_{k+1}^{\varepsilon} - f_k^{\varepsilon}\|_{L_{T_*}^{\infty} H_x^m L_{v,w}^2} \lesssim \frac{1}{2} \|f_k^{\varepsilon} - f_{k-1}^{\varepsilon}\|_{L_{T_*}^{\infty} H_x^m L_{v,w}^2}. \tag{35}$$

By working directly with equation (27) for $j = 0$, we can show by analogous derivations as in the above paragraph that the inequality

$$\|f_1^{\varepsilon} - f_0^{\varepsilon}\|_{L_T^{\infty} H_x^m L_{v,w}^2} \lesssim \frac{(1+\nu)T}{\varepsilon^2 \nu} \left(\|f_0^{\varepsilon}\|_{L_T^{\infty} H_x^m L_{v,w}^2} + \|f_0^{\varepsilon}\|_{L_T^{\infty} H_x^m L_{v,w}^2}^2 \right)$$

holds for any $T > 0$. By assumption (34) and inequality (35), we deduce that for all j that takes integer value from 2 to $k+1$, it holds that

$$\|f_j^{\varepsilon} - f_{j-1}^{\varepsilon}\|_{L_{T_*}^{\infty} H_x^m L_{v,w}^2} \lesssim \frac{1}{2^{j-1}} \|f_1^{\varepsilon} - f_0^{\varepsilon}\|_{L_{T_*}^{\infty} H_x^m L_{v,w}^2}.$$

As a result,

$$\|f_{k+1}^{\varepsilon}\|_{L_{T_*}^{\infty} H_x^m L_{v,w}^2} \lesssim \|f_0^{\varepsilon}\|_{H_x^m L_{v,w}^2} + \sum_{j=1}^{k+1} \|f_j^{\varepsilon} - f_{j-1}^{\varepsilon}\|_{L_{T_*}^{\infty} H_x^m L_{v,w}^2}$$

$$\leqslant \|f_0^{\varepsilon}\|_{H_x^m L_{v,w}^2} + \|f_1^{\varepsilon} - f_0^{\varepsilon}\|_{L_{T_*}^{\infty} H_x^m L_{v,w}^2} \cdot \sum_{j=1}^{k+1} \frac{1}{2^{j-1}}$$

$$\leqslant \|f_0^{\varepsilon}\|_{H_x^m L_{v,w}^2} + 2\|f_1^{\varepsilon} - f_0^{\varepsilon}\|_{L_{T_*}^{\infty} H_x^m L_{v,w}^2} \lesssim M(f_0^{\varepsilon}, T_*).$$

This completes the proof of the induction. By the contraction mapping principle, the sequence $\{f_j^{\varepsilon}\}_{j \in \mathbf{N}_0}$ is indeed Cauchy in $L^{\infty}([0, T_*]; H_x^m L_{v,w}^2)$. Taking the limit as $j \to \infty$, we obtain a local solution to the approximate DVBGK Boltzmann equation (12). □

Remark 8. In the proof of lemma 2, the existence of a local solution f^{ε} to the approximate DVBGK Boltzmann equation (12) is concluded by the Banach fixed point theorem. Hence, for any $t \in [0, T_*]$, it holds that

$$f^{\varepsilon} = \Lambda_{\varepsilon}(f^{\varepsilon}).$$

With the help of the local energy estimate (15) and the standard continuous induction argument, we can extend this local solution f^{ε} further to a local solution $\widetilde{f^{\varepsilon}} \in L^{\infty}([0, T_0]; H_x^m L_{v,w}^2)$ satisfying

$$\widetilde{f^{\varepsilon}} = \Lambda_{\varepsilon}\left(\widetilde{f^{\varepsilon}}\right), \quad \forall t \in [0, T_0],$$

where the existence time T_0 is a constant that only depends on the initial data $g^{0,\varepsilon}$ and m. Hence, without loss of generality, we may always assume that f^{ε} satisfies $f^{\varepsilon} = \Lambda_{\varepsilon}(f^{\varepsilon})$ whenever we consider a local solution f^{ε} constructed in lemma 2.

As a direct application of lemma 2, we have the following implication.

Corollary 9. *Let $m \in \mathbf{N}$ with $m > d$ and $\{\varepsilon_n\}_{n \in \mathbf{N}} \subset (0, 1)$ be a sequence that converges to zero as $n \to \infty$. Let $\{g^{0,\varepsilon_n}\}_{n \in \mathbf{N}}$ be a sequence of initial conditions satisfying $g^{0,\varepsilon_n} \in H_x^m L_{v,w}^2$ for any $n \in \mathbf{N}$ and*

$$M_* := \sup_{n \in \mathbf{N}} \|g^{0,\varepsilon_n}\|_{H_x^m L_{v,w}^2} < \infty.$$

Then, there exists $T_0^* = T_0^*(M_*, m) > 0$ such that for each $n \in \mathbf{N}$, there exists a unique local solution $g^{\varepsilon_n} \in L^\infty([0, T_0^*]; H_x^m L_{v,w}^2)$ to the approximate DVBGK Boltzmann equation (12) with initial condition $\Lambda_\varepsilon(g^{0,\varepsilon_n})$ satisfying $\Lambda_\varepsilon(g^{\varepsilon_n}) = g^{\varepsilon_n}$ and the local energy estimate (15). Let

$$\rho^{\varepsilon_n} = \sum_{i=0}^{n} w_i g_i^{\varepsilon_n}, \quad u^{\varepsilon_n} = \sum_{i=0}^{n} w_i v_i g_i^{\varepsilon_n}.$$

It holds that

$$\sup_{n \in \mathbf{N}} \|\rho^{\varepsilon_n}\|_{L^\infty([0,T_0^*];H^m(\mathbf{R}_x^d))} + \sup_{n \in \mathbf{N}} \|u^{\varepsilon_n}\|_{L^\infty([0,T_0^*];H^m(\mathbf{R}_x^d))} \leqslant 3(2+\nu)^{\frac{1}{2}} M_*. \quad (36)$$

By suppressing subsequences, there exist

$$g \in L^\infty\left([0, T_0^*]; H_x^m L_{v,w}^2\right) \quad \text{and} \quad (\rho, u) \in L^\infty\left([0, T_0^*]; H^m(\mathbf{R}_x^d)\right)$$

such that $g^{\varepsilon_n} \to g$ weak-$*$ in time t, weakly in $H_x^m L_{v,w}^2$ and $(\rho^{\varepsilon_n}, u^{\varepsilon_n}) \to (\rho, u)$ weak-$*$ in time t, weakly in $H^m(\mathbf{R}_x^d)$ as $n \to \infty$. In particular,

$$\rho = \sum_{i=0}^{n} w_i g_i, \quad u = \sum_{i=0}^{n} w_i v_i g_i$$

and

$$\mathcal{G}_{\mathrm{eq}}^{\varepsilon_n} - g^{\varepsilon_n} \to 0 \quad \text{in} \quad L^2\left([0, T_0^*]; H_x^m L_{v,w}^2\right) \quad \text{as} \quad n \to \infty. \quad (37)$$

Proof. By lemma 2, we can observe that if we choose

$$T_0^* \leqslant \frac{\nu}{2 C_{LE} M_*^2} \quad (38)$$

where C_{LE} is the constant introduced in lemma 6, then for any $n \in \mathbf{N}$, there exists a local solution $g^{\varepsilon_n} \in L^\infty([0, T_0^*]; H_x^m L_{v,w}^2)$ to the approximate DVBGK Boltzmann equation (12) with initial data $\Lambda_\varepsilon(g^{0,\varepsilon_n})$ satisfying $\Lambda_\varepsilon(g^{\varepsilon_n}) = g^{\varepsilon_n}$ and the local energy estimate (15) on the time interval $[0, T_0^*]$. By Plancherel's identity, for each $n \in \mathbf{N}$, we have that $\|\Lambda_\varepsilon(g^{0,\varepsilon_n})\|_{H_x^m L_{v,w}^2} \leqslant \|g^{0,\varepsilon_n}\|_{H_x^m L_{v,w}^2}$. Together with (38), we can deduce from the local energy estimate (15) that

$$\sup_{n \in \mathbf{N}, t \in [0, T_0^*]} \|g^{\varepsilon_n}(t)\|_{H_x^m L_{v,w}^2} \leqslant (2+\nu)^{\frac{1}{2}} M_*. \quad (39)$$

Estimate (36) is a direct consequence of (39).

Since $H^m(\mathbf{R}_x^d)$ is separable, $H^{-m}(\mathbf{R}_x^d)$ is also separable [6, theorem 3.26]. Since $L^1([0,T_0^*])$ is separable, $L^1([0,T_0^*];H^{-m}(\mathbf{R}_x^d))$ and $L^1([0,T_0^*];H_x^{-m}L_{v,w}^2)$ are both separable. Since the sequences

$$\left\{\|g^{\varepsilon_n}\|_{L^\infty([0,T_0^*];H_x^mL_{v,w}^2)}\right\}_{n\in\mathbf{N}},\quad \left\{\|\rho^{\varepsilon_n}\|_{L^\infty([0,T_0^*];H^m(\mathbf{R}_x^d))}\right\}_{n\in\mathbf{N}},\quad \left\{\|u^{\varepsilon_n}\|_{L^\infty([0,T_0^*];H^m(\mathbf{R}_x^d))}\right\}_{n\in\mathbf{N}}$$

are all uniformly bounded, by suppressing subsequences, we conclude by [6, corollary 3.30] that there exist $g \in H_x^m L_{v,w}^2$ and $(\rho,u) \in H^m(\mathbf{R}_x^d)$ such that $g^{\varepsilon_n} \to g$ weak-$*$ in time t, weakly in $H_x^m L_{v,w}^2$ and $(\rho^{\varepsilon_n},u^{\varepsilon_n}) \to (\rho,u)$ weak-$*$ in time t, weakly in $H^m(\mathbf{R}_x^d)$ as $n \to \infty$. Finally, the convergence (37) results from the implication of local energy inequality (15) that

$$\frac{1}{\varepsilon^2}\int_0^{T_0^*}\|\mathcal{G}_{\text{eq}}^{\varepsilon_n}(s)-g^{\varepsilon_n}(s)\|_{H_x^mL_{v,w}^2}^2\,\mathrm{d}s \leqslant (2+\nu)\|g_0^{\varepsilon_n}\|_{H_x^mL_{v,w}^2}^2 \leqslant (2+\nu)M_*^2.$$

This completes the proof of corollary 9. $\hspace{2cm}\square$

Remark 10. Following the assumption made in corollary 9, since

$$\rho^{\varepsilon_n}\overset{*}{\rightharpoonup}\rho\quad\text{and}\quad u^{\varepsilon_n}\overset{*}{\rightharpoonup}u\quad\text{in}\quad \sigma\left(L^\infty\left([0,T_0];H^m\left(\mathbf{R}_x^d\right)\right),L^1\left([0,T_0];H^{-m}\left(\mathbf{R}_x^d\right)\right)\right),$$

we have by the uniform estimate (36) that

$$\|\rho\|_{L^\infty([0,T_0];H^m(\mathbf{R}_x^d))} \leqslant \liminf_{n\to\infty}\|\rho^{\varepsilon_n}\|_{L^\infty([0,T_0];H^m(\mathbf{R}_x^d))} \lesssim (2+\nu)^{\frac{1}{2}}M_*,$$

$$\|u\|_{L^\infty([0,T_0];H^m(\mathbf{R}_x^d))} \leqslant \liminf_{n\to\infty}\|u^{\varepsilon_n}\|_{L^\infty([0,T_0];H^m(\mathbf{R}_x^d))} \lesssim (2+\nu)^{\frac{1}{2}}M_*,$$

see e.g. [6, proposition 3.13].

2.3. Local well-posedness for the approximate equation

The uniqueness and continuous dependence on initial data for the local solution can be proved by deriving the energy inequality for the difference between two local solutions for the approximate equation (12) that are well-defined on the same time interval.

Proof of lemma 2 (Uniqueness and continuity with respect to initial data). Let $\varepsilon > 0$ be fixed. Suppose that $f^\varepsilon, g^\varepsilon \in L^\infty([0,T_0];H_x^mL_{v,w}^2)$ are two local solutions for the approximate DVBGK Boltzmann equation (12) on the same isotropic lattice $(\mathcal{V},w)$ satisfying

$$\begin{cases} f^\varepsilon = \Lambda_\varepsilon(f^\varepsilon), & f^\varepsilon\big|_{t=0} = \zeta^{0,\varepsilon}, \\ g^\varepsilon = \Lambda_\varepsilon(g^\varepsilon), & g^\varepsilon\big|_{t=0} = \eta^{0,\varepsilon} \end{cases}$$

and the local energy inequality (15) respectively. Their difference $h^\varepsilon := f^\varepsilon - g^\varepsilon$ satisfies the vector equation

$$\varepsilon\partial_t h^\varepsilon + v\cdot\nabla_x h^\varepsilon = \frac{1}{\varepsilon\nu}\left(\mathcal{H}_{\text{eq}}^\varepsilon - h^\varepsilon\right),\quad h^\varepsilon\big|_{t=0} = \zeta^{0,\varepsilon}-\eta^{0,\varepsilon}, \tag{40}$$

where for any $0 \leqslant i \leqslant n$,

$$\mathcal{H}_{i,\text{eq}}^\varepsilon := \rho_h^\varepsilon + \frac{v_i\cdot u_h^\varepsilon}{c_s^2} + \frac{\varepsilon}{2c_s^4}\sum_{\alpha,\beta=1}^d \Lambda_\varepsilon\left(u_{f,\alpha}^\varepsilon u_{f,\beta}^\varepsilon - u_{g,\alpha}^\varepsilon u_{g,\beta}^\varepsilon\right)\left(v_{i,\alpha}v_{i,\beta}-c_s^2\delta_{\alpha\beta}\right)$$

with notations

$$\rho_z^\varepsilon := \sum_{i=0}^n w_i z_i^\varepsilon, \quad u_z^\varepsilon := \sum_{i=0}^n w_i v_i z_i^\varepsilon \quad \text{for} \quad z^\varepsilon \in L^\infty\left([0,T_0]; H_x^m L_{v,w}^2\right)$$

and $u_{z,\alpha}^\varepsilon$ denotes the αth component of u_z^ε for any $1 \leqslant \alpha \leqslant d$. Then, by applying ∂_x^τ with $|\tau|_s \leqslant m$ to the difference equation (40) and taking its inner product with $\partial_x^\tau h^\varepsilon$ in the $L_x^2 L_{v,w}^2$ sense, we obtain that

$$\frac{1}{2}\sum_{i=0}^n w_i \frac{\mathrm{d}}{\mathrm{d}t} \|\partial_x^\tau h_i^\varepsilon\|_{L^2(\mathbf{R}_x^d)}^2 + \frac{1}{\varepsilon^2 \nu}\sum_{i=0}^n w_i \|\partial_x^\tau \mathcal{H}_{i,\mathrm{eq}}^\varepsilon - \partial_x^\tau h_i^\varepsilon\|_{L^2(\mathbf{R}_x^d)}^2$$
$$= \frac{1}{\varepsilon^2 \nu}\sum_{i=0}^n w_i \int_{\mathbb{R}^d} \left(\partial_x^\tau \mathcal{H}_{i,\mathrm{eq}}^\varepsilon - \partial_x^\tau h_i^\varepsilon\right) \partial_x^\tau \mathcal{H}_{i,\mathrm{eq}}^\varepsilon \, \mathrm{d}x.$$

Analogously, in this case, the isotropy of $(\mathcal{V}, w)$ also guarantees that

$$\sum_{i=0}^n w_i \left(\mathcal{H}_{i,\mathrm{eq}}^\varepsilon - h_i^\varepsilon\right) = 0, \quad \sum_{i=0}^n w_i v_i \left(\mathcal{H}_{i,\mathrm{eq}}^\varepsilon - h_i^\varepsilon\right) = 0.$$

Hence, considering Plancherel's identity, we deduce that

$$\frac{1}{\varepsilon^2 \nu}\sum_{i=0}^n w_i \int_{\mathbb{R}^d} \left(\partial_x^\tau \mathcal{H}_{i,\mathrm{eq}}^\varepsilon - \partial_x^\tau h_i^\varepsilon\right) \partial_x^\tau \mathcal{H}_{i,\mathrm{eq}}^\varepsilon \, \mathrm{d}x$$

$$= \frac{1}{2c_s^4 \varepsilon \nu}\sum_{i=0}^n \sum_{\alpha,\beta=1}^d \sum_{\sigma \leqslant \tau} w_i \left(v_{i,\alpha} v_{i,\beta} - c_s^2 \delta_{\alpha\beta}\right)$$

$$\times \int_{\mathbb{R}^d} \left(\partial_x^\tau \mathcal{H}_{i,\mathrm{eq}}^\varepsilon - \partial_x^\tau h_i^\varepsilon\right) \Lambda_\varepsilon \left(\partial_x^\sigma u_{h,\alpha}^\varepsilon \partial_x^{\tau-\sigma} u_{f,\beta}^\varepsilon - \partial_x^\sigma u_{g,\alpha}^\varepsilon \partial_x^{\tau-\sigma} u_{h,\beta}^\varepsilon\right) \mathrm{d}x$$

$$\leqslant \frac{\ell_{\mathcal{V},c_s}}{2c_s^4 \varepsilon \nu}\sum_{i=0}^n w_i \|\partial_x^\tau \mathcal{H}_{i,\mathrm{eq}}^\varepsilon - \partial_x^\tau h_i^\varepsilon\|_{L^2(\mathbf{R}_x^d)} \tag{41}$$

$$\times \sum_{\alpha,\beta=1}^d \sum_{\sigma \leqslant \tau} \left(\|\partial_x^\sigma u_{h,\alpha}^\varepsilon \partial_x^{\tau-\sigma} u_{f,\beta}^\varepsilon\|_{L^2(\mathbf{R}_x^d)} + \|\partial_x^\sigma u_{g,\alpha}^\varepsilon \partial_x^{\tau-\sigma} u_{h,\beta}^\varepsilon\|_{L^2(\mathbf{R}_x^d)}\right)$$

$$\leqslant \frac{1}{2\varepsilon^2 \nu}\sum_{i=0}^n w_i \|\partial_x^\tau \mathcal{H}_{i,\mathrm{eq}}^\varepsilon - \partial_x^\tau h_i^\varepsilon\|_{L^2(\mathbf{R}_x^d)}^2$$

$$+ \frac{\ell_{\mathcal{V},c_s}^2 d^2 |\tau|_s^d}{4c_s^8 \nu} \sum_{\alpha,\beta=1}^d \sum_{\sigma \leqslant \tau} \left(\|\partial_x^\sigma u_{h,\alpha}^\varepsilon \partial_x^{\tau-\sigma} u_{f,\beta}^\varepsilon\|_{L^2(\mathbf{R}_x^d)}^2 + \|\partial_x^\sigma u_{g,\alpha}^\varepsilon \partial_x^{\tau-\sigma} u_{h,\beta}^\varepsilon\|_{L^2(\mathbf{R}_x^d)}^2\right),$$

where $\ell_{\mathcal{V},c_s} := \left(\max\limits_{1 \leqslant i \leqslant n} |v_i|\right)^2 + c_s^2$. By using proposition 4 to further estimate the right hand side of (41), we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=0}^n w_i \|\partial_x^\tau h_i^\varepsilon\|_{L^2(\mathbf{R}_x^d)}^2 + \frac{1}{2\varepsilon^2 \nu}\sum_{i=0}^n w_i \|\partial_x^\tau \mathcal{H}_{i,\mathrm{eq}}^\varepsilon - \partial_x^\tau h_i^\varepsilon\|_{L^2(\mathbf{R}_x^d)}^2$$
$$\leqslant \frac{\ell_{\mathcal{V},c_s}^2 d^4 |\tau|_s^d}{4c_s^8 \nu} \left(\|h^\varepsilon\|_{H_x^m L_{v,w}^2}^2 \|f^\varepsilon\|_{H_x^{|\tau|_s} L_{v,w}^2}^2 + \|g^\varepsilon\|_{H_x^m L_{v,w}^2}^2 \|h^\varepsilon\|_{H_x^{|\tau|_s} L_{v,w}^2}^2\right). \tag{42}$$

Summing up inequality (42) over all $\tau \in \mathbf{N}_0^d$ satisfying $|\tau|_s \leqslant m$ gives us

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h^\varepsilon\|_{H_x^m L_{v,w}^2}^2 + \frac{1}{2\varepsilon^2\nu}\|\mathcal{H}_{\mathrm{eq}}^\varepsilon - h^\varepsilon\|_{H_x^m L_{v,w}^2}^2 \\
&\leqslant \frac{\ell_{\mathcal{V},c_s}^2 d^4 m^d}{4c_s^8 \nu}\|h^\varepsilon\|_{H_x^m L_{v,w}^2}^2\left(\|f^\varepsilon\|_{H_x^m L_{v,w}^2}^2 + \|g^\varepsilon\|_{H_x^m L_{v,w}^2}^2\right).
\end{aligned}
\tag{43}
$$

Without loss of generality, we may assume that T_0, which depends on sizes of $\zeta^{0,\varepsilon}$ and $\eta^{0,\varepsilon}$, is chosen to be sufficiently small so that simultaneously

$$
\begin{cases}
\displaystyle\sup_{t\in[0,T_0]}\|f^\varepsilon(t)\|_{H_x^m L_{v,w}^2}^2 \leqslant (\nu+2)\|\zeta^{0,\varepsilon}\|_{H_x^m L_{v,w}^2}^2,\\
\displaystyle\sup_{t\in[0,T_0]}\|g^\varepsilon(t)\|_{H_x^m L_{v,w}^2}^2 \leqslant (\nu+2)\|\eta^{0,\varepsilon}\|_{H_x^m L_{v,w}^2}^2.
\end{cases}
$$

Applying the standard Gronwall's inequality to (43), we deduce that the estimate

$$
\|h^\varepsilon(t)\|_{H_x^m L_{v,w}^2}^2 \leqslant \|\zeta^{0,\varepsilon} - \eta^{0,\varepsilon}\|_{H_x^m L_{v,w}^2}^2 \, \mathrm{e}^{tc_1\left(\zeta^{0,\varepsilon},\eta^{0,\varepsilon}\right)}
$$

holds for any $t \in [0,T_0]$ where

$$
c_1\left(\zeta^{0,\varepsilon},\eta^{0,\varepsilon}\right) := c_s^{-8}\left(2^{-1}+\nu^{-1}\right)\ell_{\mathcal{V},c_s}d^4 m^d\left(\|\zeta^{0,\varepsilon}\|_{H_x^m L_{v,w}^2}^2 + \|\eta^{0,\varepsilon}\|_{H_x^m L_{v,w}^2}^2\right).
$$

Combining with the existence of the local solution, the local well-posedness of equation (12) can thus be concluded. $\qquad\qquad\square$

3. Formal derivation of the incompressible Navier–Stokes equations

Suppose that $(\mathcal{V},w)$ is an isotropic lattice associated with speed of sound c_s. Let $\varepsilon > 0$ be fixed, $m \in \mathbf{N}$ with $m > d$ and $g_0 \in H_x^m L_{v,w}^2$. By lemma 2 and corollary 9, there exists $T_0 = T_0(g_0,\mathcal{V},c_s,m,d) > 0$ sufficiently small such that the approximate DVBGK Boltzmann equation (12) admits a unique local solution $g^\varepsilon \in L^\infty\left([0,T_0];H_x^m L_{v,w}^2\right)$ satisfying $g^\varepsilon = \Lambda_\varepsilon(g^\varepsilon)$, $g^\varepsilon\big|_{t=0} = \Lambda_\varepsilon(g_0)$ and the local energy inequality

$$
\sup_{t\in[0,T_0]}\|g^\varepsilon(t)\|_{H_x^m L_{v,w}^2}^2 + \frac{1}{\varepsilon^2}\int_0^{T_0}\|\mathcal{G}_{\mathrm{eq}}^\varepsilon(t) - g^\varepsilon(t)\|_{H_x^m L_{v,w}^2}^2\,\mathrm{d}t \leqslant (2+\nu)\|g_0\|_{H_x^m L_{v,w}^2}^2.
\tag{44}
$$

Due to (14), by taking the inner product of the approximate equation (12) with 1 and v in the $L_{v,w}^2$ sense, we obtain that

$$
\begin{cases}
\displaystyle\sum_{i=0}^n \varepsilon w_i \partial_t g_i^\varepsilon + \sum_{i=0}^n w_i v_i\cdot\nabla_x g_i^\varepsilon = 0 = \varepsilon\partial_t\rho^\varepsilon + \operatorname{div}u^\varepsilon,\\
\displaystyle\sum_{i=0}^n \varepsilon w_i v_i \partial_t g_i^\varepsilon + \sum_{i=0}^n w_i v_i\left(v_i\cdot\nabla_x g_i^\varepsilon\right) = 0 = \varepsilon\partial_t u^\varepsilon + \sum_{i=0}^n w_i \nabla_x\cdot\left(v_i\otimes v_i g_i^\varepsilon\right).
\end{cases}
\tag{45}
$$

For each $0 \leqslant i \leqslant n$, we define the matrix A_i by

$$
A_i = v_i \otimes v_i - c_s^2 I
$$

with I denoting the identity matrix. By making use of the matrix A_i, for each i we have that

$$\nabla_x \cdot (v_i \otimes v_i g_i^\varepsilon) = \nabla_x \cdot (A_i g_i^\varepsilon) + c_{\mathrm{s}}^2 \nabla_x g_i^\varepsilon.$$

Furthermore, we decompose

$$\nabla_x \cdot (A_i g_i^\varepsilon) = \nabla_x \cdot \left(A_i \left(g_i^\varepsilon - \mathcal{G}_{i,\mathrm{eq}}^\varepsilon \right) \right) + \nabla_x \cdot \left(A_i \mathcal{G}_{i,\mathrm{eq}}^\varepsilon \right).$$

3.1. Derivation of the transport term $u^\varepsilon \cdot \nabla_x u^\varepsilon$

In this section, we show that the transport term $u^\varepsilon \cdot \nabla_x u^\varepsilon$ can be obtained by rewriting the summation of $\varepsilon^{-1} w_i \nabla_x \cdot (A_i \mathcal{G}_{i,\mathrm{eq}}^\varepsilon)$ in i.

Lemma 11. *It holds that*

$$\sum_{i=0}^n \frac{1}{\varepsilon} w_i \nabla_x \cdot \left(A_i \mathcal{G}_{i,\mathrm{eq}}^\varepsilon \right) = \nabla_x \cdot (u^\varepsilon \otimes u^\varepsilon) + \mathcal{R}_T^\varepsilon(x,t)$$

where $\mathcal{R}_T^\varepsilon(x,t)$ is a remainder vector depending on x and t which converges to zero in the sense of distributions, i.e. for any $\Phi \in C_{\mathrm{c}}^\infty(\mathbf{R}^d \times [0, T_0])$,

$$\left| \int_0^{T_0} \int_{\mathbf{R}^d} \mathcal{R}_T^\varepsilon(x,t) \Phi(x,t) \, \mathrm{d}x \mathrm{d}t \right| \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

More explicitly, the remainder vector $\mathcal{R}_T^\varepsilon(x,t)$ is defined by expressions (49) and (50) within the proof of this lemma.

Proof. Substituting expression (13) for $\mathcal{G}_{i,\mathrm{eq}}^\varepsilon$ directly, we have that

$$\sum_{i=0}^n \frac{1}{\varepsilon} w_i \nabla_x \cdot \left(A_i \mathcal{G}_{i,\mathrm{eq}}^\varepsilon \right) = \sum_{i=0}^n \frac{1}{\varepsilon} \nabla_x \cdot \left(w_i A_i \rho^\varepsilon + w_i A_i \frac{v_i \cdot u^\varepsilon}{c_{\mathrm{s}}^2} \right) \tag{46}$$
$$+ \frac{1}{2c_{\mathrm{s}}^4} \sum_{i=0}^n \sum_{\alpha,\beta=1}^d \nabla_x \cdot \left(w_i A_i \left(v_{i,\alpha} v_{i,\beta} - c_{\mathrm{s}}^2 \delta_{\alpha\beta} \right) \Lambda_\varepsilon \left(u_\alpha^\varepsilon u_\beta^\varepsilon \right) \right).$$

By using the second to the fourth summation condition in (4), we observe that for any $1 \leqslant \alpha, \beta \leqslant d$,

$$\sum_{i=0}^n w_i A_{i,\alpha,\beta} = \sum_{i=0}^n w_i \left(v_{i,\alpha} v_{i,\beta} - c_{\mathrm{s}}^2 \delta_{\alpha\beta} \right) = \sum_{i=0}^n w_i v_{i,\alpha} v_{i,\beta} - c_{\mathrm{s}}^2 \delta_{\alpha\beta} = 0$$

and

$$\sum_{i=0}^n \sum_{\gamma=1}^d w_i A_{i,\alpha,\beta} v_{i,\gamma} u_\gamma^\varepsilon = \sum_{i=0}^n \sum_{\gamma=1}^d w_i \left(v_{i,\alpha} v_{i,\beta} - c_{\mathrm{s}}^2 \delta_{\alpha\beta} \right) v_{i,\gamma} u_\gamma^\varepsilon$$
$$= \sum_{\gamma=1}^d u_\gamma^\varepsilon \left(\sum_{i=0}^n w_i v_{i,\alpha} v_{i,\beta} v_{i,\gamma} \right) - c_{\mathrm{s}}^2 \delta_{\alpha\beta} \sum_{\gamma=1}^d u_\gamma^\varepsilon \left(\sum_{i=0}^n w_i v_{i,\gamma} \right) = 0,$$

where the notation $A_{i,\alpha,\beta}$ represents the (α,β)-entry of the matrix A_i. Hence, it holds that

$$\sum_{i=0}^{n} \frac{1}{\varepsilon} \nabla_x \cdot \left(w_i A_i \rho^\varepsilon + w_i A_i \frac{v_i \cdot u^\varepsilon}{c_{\mathrm{s}}^2} \right) = 0. \tag{47}$$

Moreover, by using the third and the fifth summation condition in (4), we derive that for any $1 \leqslant \gamma, \zeta \leqslant d$,

$$
\begin{aligned}
\sum_{i=0}^{n} w_i A_{i,\gamma,\zeta} \left(v_{i,\alpha} v_{i,\beta} - c_{\mathrm{s}}^2 \delta_{\alpha\beta} \right) &= \sum_{i=0}^{n} w_i \left(v_{i,\gamma} v_{i,\zeta} - c_{\mathrm{s}}^2 \delta_{\gamma\zeta} \right) \left(v_{i,\alpha} v_{i,\beta} - c_{\mathrm{s}}^2 \delta_{\alpha\beta} \right) \\
&= \sum_{i=0}^{n} w_i v_{i,\alpha} v_{i,\beta} v_{i,\gamma} v_{i,\zeta} - c_{\mathrm{s}}^4 \delta_{\alpha\beta} \delta_{\gamma\zeta} = c_{\mathrm{s}}^4 \delta_{\alpha\gamma} \delta_{\beta\zeta} + c_{\mathrm{s}}^4 \delta_{\alpha\zeta} \delta_{\beta\gamma}.
\end{aligned}
\tag{48}
$$

By substituting (47) and (48) back into the right hand side of (46), for any $1 \leqslant \gamma \leqslant d$, we obtain that

$$
\begin{aligned}
\left\{ \sum_{i=0}^{n} \frac{1}{\varepsilon} w_i \nabla_x \cdot \left(A_i \mathcal{G}_{i,\mathrm{eq}}^\varepsilon \right) \right\}_\gamma &= \frac{1}{2} \sum_{\alpha,\beta,\zeta=1}^{d} \left(\delta_{\alpha\gamma} \delta_{\beta\zeta} + \delta_{\alpha\zeta} \delta_{\beta\gamma} \right) \partial_{x_\zeta} \Lambda_\varepsilon \left(u_\alpha^\varepsilon u_\beta^\varepsilon \right) \\
&= \frac{1}{2} \sum_{\beta,\zeta=1}^{d} \left(\delta_{\beta\zeta} \partial_{x_\zeta} \Lambda_\varepsilon \left(u_\gamma^\varepsilon u_\beta^\varepsilon \right) + \delta_{\beta\gamma} \partial_{x_\zeta} \Lambda_\varepsilon \left(u_\zeta^\varepsilon u_\beta^\varepsilon \right) \right) \\
&= \sum_{\zeta=1}^{d} \partial_{x_\zeta} \Lambda_\varepsilon \left(u_\zeta^\varepsilon u_\gamma^\varepsilon \right) = \left\{ \nabla_x \cdot (u^\varepsilon \otimes u^\varepsilon) \right\}_\gamma + \mathcal{R}_T^\varepsilon (x,t)_\gamma
\end{aligned}
$$

where $\mathcal{R}_T^\varepsilon(x,t)_\gamma$ is the remainder function

$$\mathcal{R}_T^\varepsilon(x,t)_\gamma := \sum_{\zeta=1}^{d} \partial_{x_\zeta} \left(\Lambda_\varepsilon \left(u_\zeta^\varepsilon u_\gamma^\varepsilon \right) - u_\zeta^\varepsilon u_\gamma^\varepsilon \right), \tag{49}$$

which converges to zero in the sense of distributions. Indeed, for any $1 \leqslant \gamma \leqslant d$ and $\Phi(x,t) \in C_{\mathrm{c}}^\infty(\mathbf{R}^d \times [0,T_0])$, we observe by remark 7 that

$$
\begin{aligned}
\int_0^{T_0} \int_{\mathbf{R}^d} \mathcal{R}_T^\varepsilon(x,t)_\gamma \Phi(x,t) \, \mathrm{d}x \, \mathrm{d}t &= -\int_0^{T_0} \int_{\mathbf{R}^d} \left(\Lambda_\varepsilon \left(u_\gamma^\varepsilon u^\varepsilon \right) - u_\gamma^\varepsilon u^\varepsilon \right) \cdot \nabla_x \Phi \, \mathrm{d}x \, \mathrm{d}t \\
&= -\int_0^{T_0} \int_{\mathbf{R}^d} u_\gamma^\varepsilon u^\varepsilon \cdot \left(\Lambda_\varepsilon \left(\nabla_x \Phi \right) - \nabla_x \Phi \right) \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned}
$$

By the Plancherel's identity, we deduce that

$$
\begin{aligned}
\left\| \Lambda_\varepsilon \left(\nabla_x \Phi \right) - \nabla_x \Phi \right\|_{H^{-m}\left(\mathbf{R}_x^2\right)}^2 &= 4\pi^2 \int_{|\xi| \geqslant \frac{1}{\varepsilon}} \left(1 + |\xi|^2 \right)^{-m} |\xi|^2 \left| \widehat{\Phi}(\xi,t) \right|^2 \mathrm{d}\xi \\
&\leqslant 4\pi^2 \int_{|\xi| \geqslant \frac{1}{\varepsilon}} \left| \widehat{\Phi}(\xi,t) \right|^2 \mathrm{d}\xi \to 0
\end{aligned}
$$

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as $\varepsilon \to 0$ for any $t \in [0, T_0]$. Using proposition 4 and inequality (36) to control $u_\gamma^\varepsilon u^\varepsilon$, we deduce by the dominated convergence theorem that

$$
\left| \int_0^{T_0} \int_{\mathbf{R}^d} u_\gamma^\varepsilon u^\varepsilon \cdot \left(\Lambda_\varepsilon \left(\nabla_x \Phi \right) - \nabla_x \Phi \right) \, \mathrm{d}x \, \mathrm{d}t \right|
$$

$$
\leqslant \int_0^{T_0} \| u_\gamma^\varepsilon u^\varepsilon \|_{H^m\left(\mathbf{R}_x^d\right)} \| \Lambda_\varepsilon \left(\nabla_x \Phi \right) - \nabla_x \Phi \|_{H^{-m}\left(\mathbf{R}_x^d\right)} \, \mathrm{d}t
$$

$$
\lesssim \| g_0 \|_{L^\infty\left([0,T_0]; H_x^m L_{v,w}^2 \right)}^2 \int_0^{T_0} \int_{|\xi| \geqslant \frac{1}{\varepsilon}} \left| \widehat{\Phi} \left(\xi, t \right) \right|^2 \mathrm{d}\xi \, \mathrm{d}t \to 0 \quad \text{as} \quad \varepsilon \to 0.
$$

Finally, we set the vector field

$$
\mathcal{R}_T^\varepsilon (x, t) := \left(\mathcal{R}_T^\varepsilon (x, t)_1, \mathcal{R}_T^\varepsilon (x, t)_2 \right). \tag{50}
$$

The proof of lemma 11 is then completed. □

3.2. Derivation of the diffusion term $\Delta_x u^\varepsilon$

In this section, we show that the diffusion term $\Delta_x u^\varepsilon$ can be obtained by rewriting the summation of $\varepsilon^{-1} w_i \nabla_x \cdot \left(A_i \left(g_i^\varepsilon - \mathcal{G}_{i,\text{eq}}^\varepsilon \right) \right)$ in i. For simplicity of notation, in the following we shall denote $g^\varepsilon - \mathcal{G}_{\text{eq}}^\varepsilon$ by $\mathcal{F}^\varepsilon$ and $g_i^\varepsilon - \mathcal{G}_{i,\text{eq}}^\varepsilon$ by $\mathcal{F}_i^\varepsilon$ for each $0 \leqslant i \leqslant n$.

Lemma 12. *It holds that*

$$
\sum_{i=0}^n \frac{1}{\varepsilon} w_i \nabla_x \cdot \left(A_i \mathcal{F}_i^\varepsilon \right) = -c_s^2 \nu \Delta_x u^\varepsilon - c_s^2 \nu \nabla_x \operatorname{div} u^\varepsilon + \mathcal{R}_D^\varepsilon (x, t),
$$

where $\mathcal{R}_D^\varepsilon (x, t)$ is a remainder vector depending on x and t which converges to zero in the sense of distributions, i.e. for any $\Phi \in C_c^\infty (\mathbf{R}^d \times [0, T_0])$,

$$
\left| \int_0^{T_0} \int_{\mathbf{R}^d} \mathcal{R}_D^\varepsilon (x, t) \, \Phi (x, t) \, \mathrm{d}x \, \mathrm{d}t \right| \to 0 \quad \text{as} \quad \varepsilon \to 0.
$$

More explicitly, the remainder vector $\mathcal{R}_D^\varepsilon (x, t)$ is defined by expression (56) within the proof of this lemma.

Proof. By rearranging the approximate equation (12), we note that

$$
\mathcal{F}_i^\varepsilon = -\varepsilon^2 \nu \partial_t g_i^\varepsilon - \varepsilon \nu v_i \cdot \nabla_x g_i^\varepsilon. \tag{51}
$$

for any $0 \leqslant i \leqslant n$. By substituting expression (51) for $\mathcal{F}_i^\varepsilon$ for each i, we rewrite

$$
\sum_{i=0}^n \frac{1}{\varepsilon} w_i \nabla_x \cdot \left(A_i \mathcal{F}_i^\varepsilon \right) = -\varepsilon \nu \sum_{i=0}^n w_i \nabla_x \cdot \left(A_i \partial_t g_i^\varepsilon \right) - \nu \sum_{i=0}^n w_i \nabla_x \cdot \left(A_i \left(v_i \cdot \nabla_x g_i^\varepsilon \right) \right).
$$

Furthermore, we rewrite again

$$
\sum_{i=0}^n w_i \nabla_x \cdot \left(A_i \left(v_i \cdot \nabla_x g_i^\varepsilon \right) \right) = \sum_{i=0}^n w_i \nabla_x \cdot \left(A_i \left(v_i \cdot \nabla_x \mathcal{F}_i^\varepsilon \right) \right) + \sum_{i=0}^n w_i \nabla_x \cdot \left(A_i \left(v_i \cdot \nabla_x \mathcal{G}_{i,\text{eq}}^\varepsilon \right) \right).
$$

What we are about to show is that the diffusion term $\Delta_x u^\varepsilon$ comes from

$$\sum_{i=0}^{n} w_i \nabla_x \cdot \left(A_i \left(v_i \cdot \nabla_x \mathcal{G}_{i,\mathrm{eq}}^\varepsilon \right) \right) \tag{52}$$

whereas

$$\varepsilon \sum_{i=0}^{n} w_i \nabla_x \cdot \left(A_i \partial_t g_i^\varepsilon \right) + \sum_{i=0}^{n} w_i \nabla_x \cdot \left(A_i \left(v_i \cdot \nabla_x \mathcal{F}_i^\varepsilon \right) \right) \to 0$$

in the sense of distributions as $\varepsilon \to 0$.

By substituting expression (13) for $\mathcal{G}_{i,\mathrm{eq}}^\varepsilon$ into (52), we observe that

$$\sum_{i=0}^{n} w_i \nabla_x \cdot \left(A_i \left(v_i \cdot \nabla_x \mathcal{G}_{i,\mathrm{eq}}^\varepsilon \right) \right)$$

$$= \sum_{i=0}^{n} w_i \nabla_x \cdot \left(A_i \left(v_i \cdot \nabla_x \rho^\varepsilon \right) \right) + \frac{1}{c_s^2} \sum_{i=0}^{n} w_i \nabla_x \cdot \left(A_i \left(v_i \cdot \nabla_x \left(v_i \cdot u^\varepsilon \right) \right) \right)$$

$$+ \frac{\varepsilon}{2 c_s^4} \sum_{i=0}^{n} w_i \nabla_x \cdot \left(\sum_{\alpha,\beta=1}^{d} A_{i,\alpha,\beta} A_i \left(v_i \cdot \nabla_x \Lambda_\varepsilon \left(u_\alpha^\varepsilon u_\beta^\varepsilon \right) \right) \right).$$

For $1 \leqslant \beta \leqslant d$, we can deduce from the second and the fourth summation condition in (4) that

$$\left\{ \sum_{i=0}^{n} w_i \nabla_x \cdot \left(A_i \left(v_i \cdot \nabla_x \rho^\varepsilon \right) \right) \right\}_\beta = \sum_{i=0}^{n} \sum_{\alpha,\gamma=1}^{d} w_i \left(v_{i,\alpha} v_{i,\beta} - c_s^2 \delta_{\alpha\beta} \right) v_{i,\gamma} \partial_{x_\alpha} \partial_{x_\gamma} \rho^\varepsilon$$

$$= \sum_{\alpha,\gamma=1}^{d} \partial_{x_\alpha} \partial_{x_\gamma} \rho^\varepsilon \left(\sum_{i=0}^{n} w_i v_{i,\alpha} v_{i,\beta} v_{i,\gamma} - c_s^2 \delta_{\alpha\beta} \sum_{i=0}^{n} w_i v_{i,\gamma} \right) = 0.$$

For $0 \leqslant i \leqslant n$, we observe that

$$h_i^\varepsilon := v_i \cdot \nabla_x \left(v_i \cdot u^\varepsilon \right) = \sum_{\alpha=1}^{d} v_{i,\alpha} \partial_{x_\alpha} \left(v_i \cdot u^\varepsilon \right) = \sum_{\alpha,\beta=1}^{d} v_{i,\alpha} v_{i,\beta} \partial_{x_\alpha} u_\beta^\varepsilon.$$

Thus, for $1 \leqslant j \leqslant d$, we have that

$$\left\{ \sum_{i=0}^{n} w_i \nabla_x \cdot \left(A_i h_i^\varepsilon \right) \right\}_j = \sum_{i=0}^{n} \sum_{k=1}^{d} w_i A_{i,k,j} \partial_{x_k} h_i^\varepsilon = \sum_{i=0}^{n} \sum_{k,\alpha,\beta=1}^{d} w_i A_{i,k,j} v_{i,\alpha} v_{i,\beta} \partial_{x_k} \partial_{x_\alpha} u_\beta^\varepsilon$$

$$\tag{53}$$

$$= \sum_{i=0}^{n} \sum_{k,\alpha,\beta=1}^{d} w_i v_{i,k} v_{i,j} v_{i,\alpha} v_{i,\beta} \partial_{x_k} \partial_{x_\alpha} u_\beta^\varepsilon - c_s^2 \sum_{i=0}^{n} \sum_{k,\alpha,\beta=1}^{d} w_i \delta_{kj} v_{i,\alpha} v_{i,\beta} \partial_{x_k} \partial_{x_\alpha} u_\beta^\varepsilon.$$

Again, by the third summation condition in (4), we deduce that

$$
\sum_{i=0}^{n}\sum_{k,\alpha,\beta=1}^{d} w_i \delta_{kj} v_{i,\alpha} v_{i,\beta} \partial_{x_k} \partial_{x_\alpha} u_\beta^\varepsilon = \sum_{i=0}^{n}\sum_{\alpha,\beta=1}^{d} w_i v_{i,\alpha} v_{i,\beta} \partial_{x_j} \partial_{x_\alpha} u_\beta^\varepsilon
$$
$$
= \sum_{\alpha,\beta=1}^{d} \left(\sum_{i=0}^{n} w_i v_{i,\alpha} v_{i,\beta} \right) \partial_{x_j} \partial_{x_\alpha} u_\beta^\varepsilon = c_{\mathrm{s}}^2 \sum_{\alpha=1}^{d} \partial_{x_j} \partial_{x_\alpha} u_\alpha^\varepsilon.
$$

$$(54)$$

On the other hand, we deduce by the fifth summation condition in (4) that

$$
\sum_{i=0}^{n}\sum_{k,\alpha,\beta=1}^{d} w_i v_{i,\alpha} v_{i,\beta} v_{i,k} v_{i,j} \partial_{x_k} \partial_{x_\alpha} u_\beta^\varepsilon = c_{\mathrm{s}}^4 \sum_{k,\alpha,\beta=1}^{d} \left(\delta_{\alpha\beta}\delta_{kj} + \delta_{\alpha k}\delta_{\beta j} + \delta_{\alpha j}\delta_{\beta k} \right) \partial_{x_k} \partial_{x_\alpha} u_\beta^\varepsilon
$$
$$
= c_{\mathrm{s}}^4 \left\{ \sum_{\alpha,\beta=1}^{d} \delta_{\alpha\beta} \partial_{x_j} \partial_{x_\alpha} u_\beta^\varepsilon + \sum_{k,\alpha=1}^{d} \delta_{\alpha k} \partial_{x_k} \partial_{x_\alpha} u_j^\varepsilon + \sum_{k,\beta=1}^{d} \delta_{\beta k} \partial_{x_k} \partial_{x_j} u_\beta^\varepsilon \right\}
$$
$$
= c_{\mathrm{s}}^4 \left\{ \sum_{\alpha=1}^{d} \partial_{x_j} \partial_{x_\alpha} u_\alpha^\varepsilon + \sum_{\alpha=1}^{d} \partial_{x_\alpha}^2 u_j^\varepsilon + \sum_{\beta=1}^{d} \partial_{x_\beta} \partial_{x_j} u_\beta^\varepsilon \right\}.
$$

$$(55)$$

Substitute (54) and (55) back into (53), we obtain that

$$
\left\{ \sum_{i=0}^{n} w_i \nabla_x \cdot \left(A_i h_i^\varepsilon \right) \right\}_j = c_{\mathrm{s}}^4 \Delta_x u_j^\varepsilon + c_{\mathrm{s}}^4 \partial_{x_j} \operatorname{div} u^\varepsilon
$$

for any $1 \leqslant j \leqslant d$.

We set

$$
\mathcal{R}_D^\varepsilon(x,t) := -\varepsilon\nu \sum_{i=0}^{n} w_i \nabla_x \cdot \left(A_i \partial_t g_i^\varepsilon \right) - \nu \sum_{i=0}^{n} w_i \nabla_x \cdot \left(A_i \left(v_i \cdot \nabla_x \mathcal{F}_i^\varepsilon \right) \right)
$$
$$
- \frac{\varepsilon\nu}{2c_{\mathrm{s}}^4} \sum_{i=0}^{n} w_i \nabla_x \cdot \left(\sum_{\alpha,\beta=1}^{d} A_{i,\alpha,\beta} A_i \left(v_i \cdot \nabla_x \Lambda_\varepsilon \left(u_\alpha^\varepsilon u_\beta^\varepsilon \right) \right) \right).
$$

$$(56)$$

It is not hard to see that $\mathcal{R}_D^\varepsilon$ converges to zero in the sense of distributions as $\varepsilon \to 0$. Indeed, we take $\Phi \in C_{\mathrm{c}}^\infty(\mathbf{R}^d \times [0, T_0])$. By the local energy estimate (44), for any $1 \leqslant \beta \leqslant d$, we have that

$$
\left| \sum_{i=0}^{n} w_i \left\{ \int_0^{T_0} \int_{\mathbf{R}^d} \nabla_x \cdot \left(A_i \left(v_i \cdot \nabla_x \mathcal{F}_i^\varepsilon \right) \right) \Phi \, dx \, dt \right\}_\beta \right|
$$
$$
= \left| \sum_{i=0}^{n} \sum_{\alpha,\gamma=1}^{d} w_i A_{i,\alpha,\beta} v_{i,\gamma} \int_0^{T_0} \int_{\mathbf{R}^d} \mathcal{F}_i^\varepsilon \partial_{x_\gamma} \partial_{x_\alpha} \Phi \, dx \, dt \right|
$$
$$
\lesssim T_0^{\frac{1}{2}} \|\mathcal{F}^\varepsilon\|_{L^2\left([0,T_0]; H_x^m L_{v,w}^2\right)} \|\nabla^2 \Phi\|_{L^\infty\left([0,T_0]; H^{-m}(\mathbf{R}_x^d)\right)}
$$
$$
\lesssim \varepsilon T_0^{\frac{1}{2}} \|g_0\|_{H_x^m L_{v,w}^2} \|\nabla^2 \Phi\|_{L^\infty\left([0,T_0]; H^{-m}(\mathbf{R}_x^d)\right)} \to 0 \quad \text{as} \quad \varepsilon \to 0.
$$

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On the other hand, by estimate (39), we have that

$$
\left| \sum_{i=0}^{n} w_i \int_0^{T_0} \int_{\mathbf{R}^d} \Phi \nabla_x \cdot \left(A_i \partial_t g_i^\varepsilon \right) \, \mathrm{d}x \, \mathrm{d}t \right| = \left| \sum_{i=0}^{n} w_i \int_0^{T_0} \int_{\mathbf{R}^d} \left(A_i^{\mathrm{T}} \partial_t g_i^\varepsilon \right) \cdot \nabla_x \Phi \, \mathrm{d}x \, \mathrm{d}t \right|
$$

$$
\lesssim \sum_{i=0}^{n} w_i \int_{\mathbf{R}^d} \left| g_{i,0}^\varepsilon \right| \left| \left(\nabla_x \Phi \right)(x,0) \right| \, \mathrm{d}x + \sum_{i=0}^{n} w_i \int_{\mathbf{R}^d} \left| g_i^\varepsilon \left(T_0 \right) \right| \left| \left(\nabla_x \Phi \right)(x,T_0) \right| \, \mathrm{d}x
$$

$$
+ \sum_{i=0}^{n} w_i \int_0^{T_0} \int_{\mathbf{R}^d} \left| g_i^\varepsilon \right| \left| \partial_t \nabla_x \Phi \right| \, \mathrm{d}x \, \mathrm{d}t
$$

$$
\lesssim \| g_0 \|_{L^\infty \left([0,T_0]; H_x^m L_{v,w}^2 \right)} \left(\| \nabla_x \Phi \|_{L^\infty \left([0,T_0]; H^{-m} \left(\mathbf{R}_x^d \right) \right)} + T_0 \| \partial_t \nabla_x \Phi \|_{L^\infty \left([0,T_0]; H^{-m} \left(\mathbf{R}_x^d \right) \right)} \right) < \infty.
$$

Analogously, for any $1 \leqslant j \leqslant d$, by using proposition 4, Plancherel's identity and estimate (39), we deduce that

$$
\left| \sum_{i=0}^{n} w_i \left\{ \int_0^{T_0} \int_{\mathbf{R}^d} \Phi \nabla_x \cdot \left(\sum_{\alpha,\beta=1}^{d} A_{i,\alpha,\beta} A_i \left(v_i \cdot \nabla_x \Lambda_\varepsilon \left(u_\alpha^\varepsilon u_\beta^\varepsilon \right) \right) \right) \, \mathrm{d}x \, \mathrm{d}t \right\}_j \right|
$$

$$
= \left| \sum_{i=0}^{n} \sum_{\alpha,\beta,\gamma,k=1}^{d} w_i A_{i,\alpha,\beta} A_{i,k,j} v_{i,\gamma} \int_0^{T_0} \int_{\mathbf{R}^d} \Lambda_\varepsilon \left(u_\alpha^\varepsilon u_\beta^\varepsilon \right) \partial_{x_k} \partial_{x_\gamma} \Phi \, \mathrm{d}x \, \mathrm{d}t \right|
$$

$$
\lesssim T_0 \| u^\varepsilon \|_{L^\infty \left([0,T_0]; H^m \left(\mathbf{R}_x^d \right) \right)}^2 \| \nabla_x^2 \Phi \|_{L^\infty \left([0,T_0]; H^{-m} \left(\mathbf{R}_x^d \right) \right)}
$$

$$
\lesssim T_0 \| g_0 \|_{L^\infty \left([0,T_0]; H_x^m L_{v,w}^2 \right)}^2 \| \nabla_x^2 \Phi \|_{L^\infty \left([0,T_0]; H^{-m} \left(\mathbf{R}_x^d \right) \right)} < \infty.
$$

This completes the proof of lemma 12. $\square$

4. The hydrodynamic limit

In this section, we further stick to the setting in section 3. Combining lemmas 11 and 12, we rewrite system (45) as

$$
\begin{cases}
\partial_t u^\varepsilon - c_{\mathrm{s}}^2 \nu \Delta_x u^\varepsilon + \nabla_x \cdot \left(u^\varepsilon \otimes u^\varepsilon \right) - c_{\mathrm{s}}^2 \nu \nabla_x \operatorname{div} u^\varepsilon + \dfrac{c_{\mathrm{s}}^2}{\varepsilon} \nabla_x \rho^\varepsilon + \mathcal{R}^\varepsilon (x,t) = 0, \\
\operatorname{div} u^\varepsilon + \varepsilon \partial_t \rho^\varepsilon = 0,
\end{cases} \tag{57}
$$

where $\mathcal{R}^\varepsilon(x,t) := \mathcal{R}_T^\varepsilon(x,t) + \mathcal{R}_D^\varepsilon(x,t)$. By the second equation of (57) and estimate (36), it is easy to observe that $\operatorname{div} u^\varepsilon \to 0$ in the sense of distributions as $\varepsilon \to 0$. More precisely, we have that

$$
\left| \int_0^{T_0} \int_{\mathbf{R}^d} \psi \operatorname{div} u^\varepsilon \, \mathrm{d}x \, \mathrm{d}t \right|
$$

$$
\leqslant \varepsilon \left| \int_{\mathbf{R}^d} \rho^\varepsilon \left(T_0 \right) \psi \left(T_0 \right) \, \mathrm{d}x \right| + \varepsilon \left| \int_{\mathbf{R}^d} \rho^\varepsilon (0) \psi (0) \, \mathrm{d}x \right| + \varepsilon \left| \int_0^{T_0} \int_{\mathbf{R}^d} \rho^\varepsilon \partial_t \psi \, \mathrm{d}x \, \mathrm{d}t \right| \tag{58}
$$

$$
\lesssim \varepsilon \| g_0 \|_{L^\infty \left([0,T_0]; H_x^m L_{v,w}^2 \right)} \left\{ \| \psi \|_{L^\infty \left([0,T_0]; H^{-m} \left(\mathbf{R}_x^d \right) \right)} + T_0^{\frac{1}{2}} \| \partial_t \psi \|_{L^2 \left([0,T_0]; H^{-m} \left(\mathbf{R}_x^d \right) \right)} \right\}
$$

for any $\psi \in L^{\infty}([0, T_0]; H^{-m}(\mathbf{R}_x^d))$ satisfying $\partial_t \psi \in L^2([0, T_0]; H^{-m}(\mathbf{R}_x^d))$. Since $u^{\varepsilon} \to u$ weak-$*$ in t and weakly in $H^m(\mathbf{R}_x^d)$, we have that

$$\int_0^{T_0} \int_{\mathbf{R}^d} \psi \operatorname{div}(u^{\varepsilon} - u) \, \mathrm{d}x \mathrm{d}t = -\int_0^{T_0} \int_{\mathbf{R}^d} (u^{\varepsilon} - u) \cdot \nabla_x \psi \, \mathrm{d}x \mathrm{d}t \to 0 \tag{59}$$

as $\varepsilon \to 0$ for any $\psi \in L^1([0, T_0]; H^1(\mathbf{R}_x^d))$ as $H^1(\mathbf{R}_x^d) \hookrightarrow H^{-m}(\mathbf{R}_x^d)$. Since T_0 is finite, by combining (58) and (59), we obtain that

$$\int_0^{T_0} \int_{\mathbf{R}^d} \psi \operatorname{div} u \, \mathrm{d}x \mathrm{d}t = 0 = \int_0^{T_0} \int_{\mathbf{R}^d} u \cdot \nabla_x \psi \, \mathrm{d}x \mathrm{d}t \tag{60}$$

for any $\psi \in L^{\infty}([0, T_0]; H^1(\mathbf{R}_x^d))$ that satisfies $\partial_t \psi \in L^2([0, T_0]; H^1(\mathbf{R}_x^d))$.

4.1. Application of the L² Helmholtz projection to equation (57)

In order to get rid of the term $\nabla_x \rho^{\varepsilon}$ whose coefficient is ε^{-1} in the first equation of (57), we consider the Helmholtz decomposition for $L^2(\mathbf{R}_x^d)$. Let us recall that for any $h \in L^2(\mathbf{R}_x^d)$, there exists a unique decomposition $h = h_0 + \nabla_x \pi$ such that

$$h_0 \in L_{\sigma}^2\left(\mathbf{R}_x^d\right) := \left\{ f \in L^2\left(\mathbf{R}_x^d\right) \mid \operatorname{div} f = 0 \right\},$$
$$\nabla_x \pi \in G^2\left(\mathbf{R}_x^d\right) := \left\{ \nabla_x \pi \in L^2\left(\mathbf{R}_x^d\right) \mid \pi \in L_{\mathrm{loc}}^2\left(\mathbf{R}_x^d\right) \right\}.$$

Moreover, it holds that

$$\|h_0\|_{L^2\left(\mathbf{R}_x^d\right)} + \|\nabla_x \pi\|_{L^2\left(\mathbf{R}_x^d\right)} \leqslant 2\|h\|_{L^2\left(\mathbf{R}_x^d\right)}. \tag{61}$$

The Helmholtz projection, denoted by $\mathbb{P}$, is the projection that maps h to h_0, i.e. we have that $\mathbb{P}(h) = h_0$ for $h \in L^2(\mathbf{R}_x^d)$. Furthermore, the Helmholtz projection $\mathbb{P}$ satisfies the properties

$$\mathbb{P}(h_0) = h_0 \quad \forall h_0 \in L_{\sigma}^2\left(\mathbf{R}_x^d\right) \quad \text{and} \quad \mathbb{P}(\nabla_x \pi) = 0 \quad \forall \nabla_x \pi \in G^2\left(\mathbf{R}_x^d\right). \tag{62}$$

Let $\mathbb{Q} := \mathbb{I} - \mathbb{P}$ where $\mathbb{I}$ denotes the identity operator. Applying the Helmholtz projection $\mathbb{P}$ to the first equation of (57), we observe by the second property of (62) that

$$\partial_t \mathbb{P}(u^{\varepsilon}) - c_s^2 \nu \Delta_x \mathbb{P}(u^{\varepsilon}) + \mathbb{P}(\nabla_x \cdot (u^{\varepsilon} \otimes u^{\varepsilon})) + \mathbb{P}(\mathcal{R}^{\varepsilon}) = 0.$$

Proposition 13. *Let $h \in H^k(\mathbf{R}_x^d)$ with $k \in \mathbf{N}$ satisfying $k \geqslant 1$. Let $h = h_0 + \nabla_x \pi$ be the Helmholtz decomposition of h in $L^2(\mathbf{R}_x^d)$. For any $\tau \in \mathbf{N}_0^d$ with $|\tau|_s \leqslant k$, the unique Helmholtz decomposition of $\partial_x^{\tau} h$ in $L^2(\mathbf{R}_x^d)$ is given by $\partial_x^{\tau} h = \partial_x^{\tau} h_0 + \nabla_x(\partial_x^{\tau} \pi)$. Moreover, it holds that*

$$\|\partial_x^{\tau} h_0\|_{L^2\left(\mathbf{R}_x^d\right)} + \|\nabla_x(\partial_x^{\tau} \pi)\|_{L^2\left(\mathbf{R}_x^d\right)} \leqslant 2\|\partial_x^{\tau} h\|_{L^2\left(\mathbf{R}_x^d\right)}.$$

Proof. Let $\tau \in \mathbf{N}_0^d$ satisfying $|\tau|_s \leqslant k$. Since we are working in the whole space $\mathbf{R}^d$, the L^2 Helmholtz projection $\mathbb{P}$ is explicitly defined, i.e. for $1 \leqslant i,j \leqslant d$, the (i,j)-entry of $\mathbb{P}$ is given by $\mathbb{P}_{ij} := \delta_{ij} + R_i R_j$ where R_i and R_j denote the ith and jth component of the Riesz transform, respectively. Hence, it can be easily observed that the differentiation ∂_x commutes with $\mathbb{P}$ and $\partial_x^{\tau} h_0 = \partial_x^{\tau} \mathbb{P}(h) = \mathbb{P}(\partial_x^{\tau} h)$. The L^2 boundedness of $\mathbb{P}$ can be proved by working with the explicit formula for $\mathbb{P}$. As a result, by estimate (61) we have that

$$\|\partial_x^{\tau} h_0\|_{L^2\left(\mathbf{R}_x^d\right)} = \|\mathbb{P}(\partial_x^{\tau} h)\|_{L^2\left(\mathbf{R}_x^d\right)} \lesssim \|\partial_x^{\tau} h\|_{L^2\left(\mathbf{R}_x^d\right)},$$

i.e. $\partial_x^\tau h_0 \in L^2(\mathbf{R}_x^d)$. Note that $\partial_x^\tau h_0$ is divergence free. Analogously, by the L^2 boundedness of $\mathbb{I} - \mathbb{P}$ and estimate (61), we have that

$$\left\| \partial_x^\tau (\nabla_x \pi) \right\|_{L^2(\mathbf{R}^d)} = \left\| \partial_x^\tau h - \mathbb{P}(\partial_x^\tau h) \right\|_{L^2(\mathbf{R}_x^d)} \lesssim \left\| \partial_x^\tau h \right\|_{L^2(\mathbf{R}_x^d)},$$

i.e. $\nabla_x(\partial_x^\tau \pi) \in L^2(\mathbf{R}_x^d)$. By noting that $\partial_x^\tau \pi \in L^2(\mathbf{R}_x^d)$ if $\tau \in \mathbf{N}_0^d$ satisfies $1 \leqslant |\tau|_s \leqslant k$, we conclude that $\partial_x^\tau h = \partial_x^\tau h_0 + \nabla_x(\partial_x^\tau \pi)$ is indeed the unique Helmholtz decomposition of $\partial_x^\tau h$ in $L^2(\mathbf{R}^d)$. $\qquad\square$

Combining proposition 13 with corollary 9, we observe that the sequence $\{\mathbb{P}(u^\varepsilon)\}_\varepsilon$ is uniformly bounded in $L^\infty([0, T_0]; H^m(\mathbf{R}_x^d))$. Hence, by suppressing subsequences again, it can be concluded that there exists $\widetilde{u} \in L^\infty([0, T_0]; H^m(\mathbf{R}_x^d))$ such that $\mathbb{P}(u^\varepsilon) \to \widetilde{u}$ where the convergence is weak-$*$ in time t and weakly in $H^m(\mathbf{R}_x^d)$, i.e. for any $\Psi \in L^1([0, T_0]; H^{-m}(\mathbf{R}_x^d))$, it holds that

$$\int_0^{T_0} \int_{\mathbf{R}^d} \mathbb{P}(u^\varepsilon) \cdot \Psi \, dx dt \to \int_0^{T_0} \int_{\mathbf{R}^d} \widetilde{u} \cdot \Psi \, dx dt \quad \text{as} \quad \varepsilon \to 0. \tag{63}$$

On the other hand, for any $\Psi \in L^1([0, T_0]; L^2(\mathbf{R}_x^d))$, it holds that

$$\int_0^{T_0} \int_{\mathbf{R}^d} \mathbb{P}(u^\varepsilon - u) \cdot \Psi \, dx dt = \int_0^{T_0} \int_{\mathbf{R}^d} (u^\varepsilon - u) \cdot \mathbb{P}(\Psi) \, dx dt \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{64}$$

Combining convergence statements (63) and (64), we deduce that

$$\int_0^{T_0} \int_{\mathbf{R}^d} \widetilde{u} \cdot \Psi \, dx dt = \int_0^{T_0} \int_{\mathbf{R}^d} \mathbb{P}(u) \cdot \Psi \, dx dt, \quad \forall \, \Psi \in L^1\left([0, T_0]; L^2(\mathbf{R}_x^d)\right). \tag{65}$$

We next prove that with property (60), the limit point u is actually divergence free, i.e. in this case we have $\mathbb{P}(u) = u$ a.e.

Lemma 14. *Suppose that $u \in L^\infty([0, T_0]; L^2(\mathbf{R}_x^d))$ satisfies*

$$\int_0^{T_0} \int_{\mathbf{R}^d} u \cdot \nabla_x \psi \, dx dt = 0, \quad \forall \, \psi \in W^{1,\infty}\left([0, T_0]; H^1(\mathbf{R}_x^d)\right).$$

Then, for any $\Psi \in C^\infty([0, T_0]; C_c^\infty(\mathbf{R}_x^d))$, it holds that

$$\int_0^{T_0} \int_{\mathbf{R}^d} \mathbb{P}(u) \cdot \Psi \, dx dt = \int_0^{T_0} \int_{\mathbf{R}^d} u \cdot \Psi \, dx dt.$$

Proof. Let $\Psi \in C^\infty([0, T_0]; C_c^\infty(\mathbf{R}_x^d))$. For each $t \in [0, T_0]$, let $\Psi(x, t) = \Psi_0(x, t) + (\nabla_x \Pi)(x, t)$ be the Helmholtz decomposition of $\Psi(\cdot, t)$ in $L^2(\mathbf{R}_x^d)$ where $\Pi(\cdot, t) = \Delta^{-1} \operatorname{div} \Psi(\cdot, t)$. By integration by parts, we can easily observe that for each $t \in [0, T_0]$, $\Pi(\cdot, t)$ is indeed a Riesz potential of $\Psi(\cdot, t)$. Since we are considering test function Ψ, there exists a constant $C(2, d)$, depending only on 2 and d, such that

$$\left\| \Pi(\cdot, t) \right\|_{L^2(\mathbf{R}_x^d)} \leqslant C(2, d) \left\| \Psi(\cdot, t) \right\|_{L^{\frac{2d}{d+2}}(\mathbf{R}_x^d)} < \infty, \tag{66}$$

see e.g. [29, chapter V theorem 1]. Combining (66) with estimate (61) for the Helmholtz decomposition, we deduce that $\Pi \in L^\infty([0, T_0]; H^1(\mathbf{R}_x^d))$. Moreover, since the operator Δ^{-1} div is linear, we have that

$$\frac{\Pi(x, t+h) - \Pi(x, t)}{h} = \Delta^{-1} \operatorname{div} \left(\frac{\Psi(x, t+h) - \Psi(x, t)}{h} \right) \tag{67}$$

for any $t, t+h \in (0, T_0)$. We next define

$$\varphi_*(x) := \sup_{t \in [0, T_0]} \left| (\partial_t \Psi)(x, t) \right|, \quad \forall x \in \mathbf{R}^d.$$

By the mean value theorem, for any $t, t+h \in (0, T_0)$, it holds that

$$\left| \Psi(x, t+h) - \Psi(x, t) \right| \leqslant h \varphi_*(x). \tag{68}$$

By integration by parts and (68), we can deduce that for any $x \in \mathbf{R}^d$ and $t, t+h \in (0, T_0)$,

$$\left| \Delta^{-1} \operatorname{div} \left(\frac{\Psi(x, t+h) - \Psi(x, t)}{h} \right) \right| \lesssim \int_{\mathbf{R}^d} \frac{1}{|x-y|^{d-1}} \left| \frac{\Psi(y, t+h) - \Psi(y, t)}{h} \right| dy$$

$$\leqslant \int_{\mathbf{R}^d} \frac{1}{|x-y|^{d-1}} |\varphi_*(y)| \, dy =: I_1(|\varphi_*|)(x).$$

Considering that Ψ is smooth with respect to both x and t and has compact support in $\mathbf{R}^d$ with respect to x, it can be easily observed that $\varphi_* \in L^2(\mathbf{R}_x^d) \cap L^\infty(\mathbf{R}_x^d)$ also has compact support. Hence, by analogous reason as (66), we have that $I_1(|\varphi_*|) \in L^2(\mathbf{R}_x^d)$. Then, by taking the limit $h \to 0$ for equality (67), we can conclude by the dominated convergence theorem that

$$(\partial_t \Pi)(x, t) = \Delta^{-1} \operatorname{div}(\partial_t \Psi), \quad \forall (x, t) \in \mathbf{R}^d \times [0, T_0].$$

Furthermore, since each component of $\nabla \Delta^{-1}$ div is nothing but a linear combination of $R_i R_j$ ($1 \leqslant i, j \leqslant d$), we observe that $\nabla \Delta^{-1}$ div is bounded in $L^2(\mathbf{R}_x^d)$. Thus, we obtain that $\partial_t \Pi \in L^\infty([0, T_0]; H^1(\mathbf{R}_x^d))$. Using (60), we can finally deduce that for any $\Psi \in C^\infty([0, T_0]; C_c^\infty(\mathbf{R}_x^d))$, it holds that

$$\int_0^{T_0} \int_{\mathbf{R}^d} \mathbb{P}(u) \cdot \Psi \, dx \, dt = \int_0^{T_0} \int_{\mathbf{R}^d} \mathbb{P}(u) \cdot \Psi_0 \, dx \, dt = \int_0^{T_0} \int_{\mathbf{R}^d} u \cdot \Psi_0 \, dx \, dt = \int_0^{T_0} \int_{\mathbf{R}^d} u \cdot \Psi \, dx \, dt.$$

This completes the proof of lemma 14. $\qquad\square$

Combining lemma 14 with the convergence statement (63) and equality (65), we establish the convergence of $\mathbb{P}(u^\varepsilon)$ to u in the sense of distributions. In fact, the convergence of $\mathbb{P}(u^\varepsilon)$ to u can be proved to be stronger than this weak sense.

Lemma 15. *Suppressing subsequences, $\mathbb{P}(u^\varepsilon) \to u$ strongly in $C([0, T_0]; H^{m-1}(\mathbf{R}_x^d))$, i.e. it holds that*

$$\|\mathbb{P}(u^\varepsilon) - u\|_{L^\infty\left([0, T_0]; H^{m-1}(\mathbf{R}_x^d)\right)} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Proof. The second equation of (45) can be written as

$$\partial_t u^\varepsilon + \frac{1}{\varepsilon} \nabla_x \cdot \left(\sum_{i=0}^n w_i A_i \mathcal{F}_i^\varepsilon \right) + \frac{c_s^2}{\varepsilon} \nabla_x \rho^\varepsilon = -\nabla_x \cdot (u^\varepsilon \otimes u^\varepsilon) - \mathcal{R}_T^\varepsilon. \tag{69}$$

where $\mathcal{F}_i^\varepsilon = g_i^\varepsilon - \mathcal{G}_{i,\mathrm{eq}}^\varepsilon$ for every $0 \leqslant i \leqslant n$. Let $\tau \in \mathbf{N}_0^d$ with $|\tau|_s \leqslant m - 1$. Applying ∂_x^τ and the Helmholtz projection $\mathbb{P}$ to equation (69), we obtain

$$\partial_t \mathbb{P}\left(\partial_x^\tau u^\varepsilon\right) + \frac{1}{\varepsilon} \mathbb{P}\left(\nabla_x \cdot \left(\sum_{i=0}^n w_i A_i \partial_x^\tau \mathcal{F}_i^\varepsilon \right) \right)$$
$$= -\sum_{\sigma \leqslant \tau} \mathbb{P}\left(\left(\partial_x^\sigma u^\varepsilon \cdot \nabla_x\right) \partial_x^{\tau-\sigma} u^\varepsilon + \partial_x^\sigma u^\varepsilon \,\mathrm{div}\left(\partial_x^{\tau-\sigma} u^\varepsilon\right) \right) - \mathbb{P}\left(\partial_x^\tau \mathcal{R}_T^\varepsilon\right). \tag{70}$$

Integrating equation (70) over the time interval $[t_1, t_2] \subseteq [0, T_0]$ and then taking its inner product with $\mathbb{P}\left(\partial_x^\tau u^\varepsilon(t_2)\right) - \mathbb{P}\left(\partial_x^\tau u^\varepsilon(t_1)\right)$ in the L_x^2 sense, we have that

$$\left\| \mathbb{P}\left(\partial_x^\tau u^\varepsilon(t_2)\right) - \mathbb{P}\left(\partial_x^\tau u^\varepsilon(t_1)\right) \right\|_{L^2\left(\mathbf{R}_x^d\right)}$$

$$= -\frac{1}{\varepsilon} \int_{t_1}^{t_2} \int_{\mathbf{R}^d} \left(\nabla_x \cdot \left(\sum_{i=0}^n w_i A_i \partial_x^\tau \mathcal{F}_i^\varepsilon \right) \right) \cdot \mathbb{P}\left(\partial_x^\tau u^\varepsilon(t_2) - \partial_x^\tau u^\varepsilon(t_1)\right) \, \mathrm{d}x \, \mathrm{d}t$$

$$- \sum_{\sigma \leqslant \tau} \int_{t_1}^{t_2} \int_{\mathbf{R}^d} \left(\partial_x^\sigma u^\varepsilon \cdot \nabla_x\right) \partial_x^{\tau-\sigma} u^\varepsilon \cdot \mathbb{P}\left(\partial_x^\tau u^\varepsilon(t_2) - \partial_x^\tau u^\varepsilon(t_1)\right) \, \mathrm{d}x \, \mathrm{d}t \tag{71}$$

$$- \sum_{\sigma \leqslant \tau} \int_{t_1}^{t_2} \int_{\mathbf{R}^d} \mathrm{div}\left(\partial_x^{\tau-\sigma} u^\varepsilon\right) \partial_x^\sigma u^\varepsilon \cdot \mathbb{P}\left(\partial_x^\tau u^\varepsilon(t_2) - \partial_x^\tau u^\varepsilon(t_1)\right) \, \mathrm{d}x \, \mathrm{d}t$$

$$- \int_{t_1}^{t_2} \int_{\mathbf{R}^d} \partial_x^\tau \mathcal{R}_T^\varepsilon \cdot \mathbb{P}\left(\partial_x^\tau u^\varepsilon(t_2) - \partial_x^\tau u^\varepsilon(t_1)\right) \, \mathrm{d}x \, \mathrm{d}t = (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}).$$

For any $t \in [t_1, t_2]$, we can deduce simply by Hölder's inequality that

$$\left| \sum_{\alpha,\beta=1}^d \int_{\mathbf{R}^d} \left(\sum_{i=0}^n w_i A_{i,\alpha,\beta} \partial_{x_\alpha} \partial_x^\tau \mathcal{F}_i^\varepsilon \right) \mathbb{P}\left(\partial_x^\tau u^\varepsilon(t_2) - \partial_x^\tau u^\varepsilon(t_1)\right)_\beta \, \mathrm{d}x \right|$$
$$\lesssim \left\| \nabla_x \partial_x^\tau \mathcal{F}^\varepsilon(t) \right\|_{L^2\left(\mathbf{R}_x^d\right)} \left\| \partial_x^\tau u^\varepsilon \right\|_{L^\infty\left([0,T_0];L^2\left(\mathbf{R}_x^d\right)\right)}.$$

Hence, by the local energy inequality (44), we have that

$$|(\mathrm{I})| \lesssim \frac{1}{\varepsilon} \left(\int_{t_1}^{t_2} \left\| \nabla_x \partial_x^\tau \mathcal{F}^\varepsilon(t) \right\|_{L^2\left(\mathbf{R}_x^d\right)}^2 \, \mathrm{d}t \right)^{\frac{1}{2}} (t_2 - t_1)^{\frac{1}{2}} \left\| \partial_x^\tau u^\varepsilon \right\|_{L^\infty\left([0,T_0];L^2\left(\mathbf{R}_x^d\right)\right)}$$
$$\lesssim (2 + \nu)^{\frac{1}{2}} (t_2 - t_1)^{\frac{1}{2}} \left\| g_0 \right\|_{H_x^m L_{v,w}^2} \left\| \partial_x^\tau g^\varepsilon \right\|_{L^\infty\left([0,T_0];L_x^2 L_{v,w}^2\right)}.$$

As we have already seen in the proof of proposition 4, for $m > d$ and $\sigma \in \mathbf{N}_0^d$ satisfying $\sigma \leqslant \tau$, either $|\sigma|_s$ or $|\tau|_s - |\sigma|_s + 1$ is less than or equal to $m - \left[\frac{d}{2}\right] - 1$. Hence, we may assume without

loss of generality that $|\sigma|_s \leqslant m - [\frac{d}{2}] - 1$. In this case, for $t \in [t_1, t_2]$, we have that

$$
\left| \sum_{\alpha,\beta=1}^{d} \int_{\mathbf{R}^d} \left(\partial_x^\sigma u_\alpha^\varepsilon\right) \left(\partial_{x_\alpha} \partial_x^{\tau-\sigma} u_\beta^\varepsilon\right) \mathbb{P}\left(\partial_x^\tau u^\varepsilon(t_2) - \partial_x^\tau u^\varepsilon(t_1)\right)_\beta \mathrm{d}x \right|
$$
$$
\lesssim \|u^\varepsilon\|_{L^\infty\left([0,T_0];H^m\left(\mathbf{R}_x^d\right)\right)}^2 \|\partial_x^\tau u^\varepsilon\|_{L^\infty\left([0,T_0];L^2\left(\mathbf{R}_x^d\right)\right)}.
$$

Hence, by the local energy inequality (44), we deduce that

$$
|(\mathrm{II})| \lesssim (2+\nu)(t_2 - t_1) \|g_0\|_{H_x^m L_{v,w}^2}^2 \|\partial_x^\tau g^\varepsilon\|_{L^\infty\left([0,T_0];L_x^2 L_{v,w}^2\right)}. \tag{72}
$$

Note that $|(\mathrm{III})|$ can be estimated in exactly the same way as $|(\mathrm{II})|$, i.e. estimate (72) holds for $|(\mathrm{III})|$. Since

$$
\left| \sum_{\alpha,\beta=1}^{d} \int_{\mathbf{R}^d} \partial_x^\tau \partial_{x_\beta} \left(\Lambda_\varepsilon\left(u_\alpha^\varepsilon u_\beta^\varepsilon\right) - u_\alpha^\varepsilon u_\beta^\varepsilon\right) \mathbb{P}\left(\partial_x^\tau u^\varepsilon(t_2) - \partial_x^\tau u^\varepsilon(t_1)\right)_\alpha \mathrm{d}x \right|
$$
$$
\lesssim \left\|\mathbb{P}\left(\partial_x^\tau u^\varepsilon(t_2) - \partial_x^\tau u^\varepsilon(t_1)\right)\right\|_{L^2\left(\mathbf{R}_x^d\right)} \sum_{\beta=1}^{d} \left\{ \|\partial_x^\tau \partial_{x_\beta} \Lambda_\varepsilon\left(u^\varepsilon u_\beta^\varepsilon\right)\|_{L^2\left(\mathbf{R}_x^d\right)} + \|\partial_x^\tau \partial_{x_\beta}\left(u^\varepsilon u_\beta^\varepsilon\right)\|_{L^2\left(\mathbf{R}_x^d\right)} \right\}
$$
$$
\lesssim \|u^\varepsilon\|_{L^\infty\left([0,T_0];H^m\left(\mathbf{R}_x^d\right)\right)}^2 \|\partial_x^\tau u^\varepsilon\|_{L^\infty\left([0,T_0];L^2\left(\mathbf{R}_x^d\right)\right)}
$$

for any $t \in [t_1, t_2]$, estimate (72) holds for $|(\mathrm{IV})|$ analogously.

By summing up (71) over all $\tau \in \mathbf{N}_0^d$ with $|\tau|_s \leqslant m-1$, we deduce by the local energy inequality (44) that

$$
\left\|\mathbb{P}\left(u^\varepsilon(t_2)\right) - \mathbb{P}\left(u^\varepsilon(t_1)\right)\right\|_{H^{m-1}\left(\mathbf{R}_x^d\right)} \lesssim (2+\nu)^{\frac{3}{2}} \left(1 + \sqrt{T_0}\right)(t_2 - t_1)^{\frac{1}{2}} \|g_0\|_{H_x^m L_{v,w}^2}^3.
$$

Therefore, this shows that $\{\mathbb{P}(u^\varepsilon)\}_\varepsilon \subset C([0,T_0];H^{m-1}(\mathbf{R}_x^d))$ and $\left\{\left\|\mathbb{P}\left(u^\varepsilon\right)(t)\right\|_{H^{m-1}\left(\mathbf{R}_x^d\right)}\right\}_\varepsilon$ is equi-continuous in time t. By the Arzelà–Ascoli theorem, we conclude by suppressing subsequences that there exists $u_* \in C([0,T_0];H^{m-1}(\mathbf{R}_x^d))$ such that

$$
\|\mathbb{P}(u^\varepsilon) - u_*\|_{L^\infty\left([0,T_0];H^{m-1}\left(\mathbf{R}_x^d\right)\right)} \to 0 \quad \text{as} \quad \varepsilon \to 0.
$$

By splitting $u_* - u$ into $(u_* - \mathbb{P}(u^\varepsilon)) + (\mathbb{P}(u^\varepsilon) - u)$, we can deduce by lemma 14 that

$$
\int_0^{T_0} \int_{\mathbf{R}^d} (u_* - u) \cdot \Psi \, \mathrm{d}x \mathrm{d}t = 0, \quad \forall\, \Psi \in C^\infty\left([0,T_0];C_c^\infty\left(\mathbf{R}_x^d\right)\right).
$$

Finally, by the fundamental lemma of the calculus of variations, we conclude that $u_* = u$ a.e. in $\mathbf{R}^d \times [0, T_0]$. This completes the proof of lemma 15. $\qquad\square$

4.2. Convergence to the incompressible Navier–Stokes equations

Finally, we decompose

$$\mathbb{P}\left(\nabla_x \cdot (u^\varepsilon \otimes u^\varepsilon)\right) = \mathbb{P}\left(\nabla_x \cdot (\mathbb{P}(u^\varepsilon) \otimes \mathbb{P}(u^\varepsilon))\right) + \mathcal{R}_{\mathbb{P}}(u^\varepsilon)$$

with

$$\mathcal{R}_{\mathbb{P}}(u^\varepsilon) := \mathbb{P}\left(\nabla_x \cdot (\mathbb{P}(u^\varepsilon) \otimes \mathbb{Q}(u^\varepsilon))\right) + \mathbb{P}\left(\nabla_x \cdot (\mathbb{Q}(u^\varepsilon) \otimes \mathbb{P}(u^\varepsilon))\right) + \mathbb{P}\left(\nabla_x \cdot (\mathbb{Q}(u^\varepsilon) \otimes \mathbb{Q}(u^\varepsilon))\right).$$

Let $C_{c,\sigma}^\infty(\mathbf{R}^d) := \{w \in C_c^\infty(\mathbf{R}^d) \mid \mathrm{div}\, w = 0 \text{ in } \mathbf{R}^d\}$.

Lemma 16. *For any* $\Phi \in C^\infty([0,T_0]; C_{c,\sigma}^\infty(\mathbf{R}_x^d))$, *it holds that*

$$\left| \int_0^{T_0} \int_{\mathbf{R}^d} \mathcal{R}_{\mathbb{P}}(u^\varepsilon) \cdot \Phi \, \mathrm{d}x\mathrm{d}t \right| \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Proof. Since the test function Φ we consider is already divergence free,

$$\int_0^{T_0} \int_{\mathbf{R}^d} \mathbb{P}\left(\nabla_x \cdot (\mathbb{P}(u^\varepsilon) \otimes \mathbb{Q}(u^\varepsilon))\right) \cdot \Phi \, \mathrm{d}x\mathrm{d}t = \int_0^{T_0} \int_{\mathbf{R}^d} \mathbb{P}(u^\varepsilon) \otimes \mathbb{Q}(u^\varepsilon) : \nabla_x \Phi \, \mathrm{d}x\mathrm{d}t.$$

We further decompose

$$\int_0^{T_0} \int_{\mathbf{R}^d} \mathbb{P}(u^\varepsilon) \otimes \mathbb{Q}(u^\varepsilon) : \nabla_x \Phi \, \mathrm{d}x\mathrm{d}t = \int_0^{T_0} \int_{\mathbf{R}^d} (\mathbb{P}(u^\varepsilon) - u) \otimes \mathbb{Q}(u^\varepsilon) : \nabla_x \Phi \, \mathrm{d}x\mathrm{d}t$$
$$+ \int_0^{T_0} \int_{\mathbf{R}^d} u \otimes \mathbb{Q}(u^\varepsilon) : \nabla_x \Phi \, \mathrm{d}x\mathrm{d}t.$$

By lemma 15, we deduce that

$$\left| \int_0^{T_0} \int_{\mathbf{R}^d} (\mathbb{P}(u^\varepsilon) - u) \otimes \mathbb{Q}(u^\varepsilon) : \nabla_x \Phi \, \mathrm{d}x\mathrm{d}t \right|$$
$$\leqslant \int_0^{T_0} \left\| \mathbb{P}(u^\varepsilon)(t) - u(t) \right\|_{L^\infty(\mathbf{R}_x^d)} \left\| \mathbb{Q}(u^\varepsilon)(t) \right\|_{L^2(\mathbf{R}_x^d)} \left\| \nabla_x \Phi \right\|_{L^2(\mathbf{R}_x^d)} \, \mathrm{d}t$$
$$\lesssim T_0 \left\| \mathbb{P}(u^\varepsilon) - u \right\|_{L^\infty([0,T_0];H^2(\mathbf{R}_x^d))} \left\| g_0 \right\|_{H_x^m L_{v,w}^2} \left\| \nabla_x \Phi \right\|_{L^\infty([0,T_0];L^2(\mathbf{R}_x^d))} \to 0$$

as $\varepsilon \to 0$. Since lemma 14 says that $\mathbb{Q}(u) = 0$ a.e. in $\mathbf{R}^d \times [0,T_0]$, we have that

$$\int_0^{T_0} \int_{\mathbf{R}^d} u \otimes \mathbb{Q}(u^\varepsilon) : \nabla_x \Phi \, \mathrm{d}x\mathrm{d}t = \int_0^{T_0} \int_{\mathbf{R}^d} u \otimes \mathbb{Q}(u^\varepsilon - u) : \nabla_x \Phi \, \mathrm{d}x\mathrm{d}t$$
$$= \int_0^{T_0} \int_{\mathbf{R}^d} (u^\varepsilon - u) \cdot \mathbb{Q}(u \cdot \nabla_x \Phi) \, \mathrm{d}x\mathrm{d}t.$$

Noting that $u \cdot \nabla_x \Phi \in L^\infty([0,T_0];H^m(\mathbf{R}_x^d))$ and $H^m(\mathbf{R}_x^d) \hookrightarrow H^{-m}(\mathbf{R}_x^d)$, we can then deduce from the weak convergence of u^ε to u in $L^\infty([0,T_0];H^m(\mathbf{R}_x^d))$ that

$$\int_0^{T_0} \int_{\mathbf{R}^d} (u^\varepsilon - u) \cdot \mathbb{Q}(u \cdot \nabla_x \Phi) \, \mathrm{d}x\mathrm{d}t \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

On the other hand, by using lemma 11, we can further deduce from system (45) that

$$
\begin{cases}
\varepsilon \partial_t \rho^\varepsilon + \operatorname{div} \mathbb{Q}(u^\varepsilon) = \quad 0, \\
\varepsilon \partial_t \mathbb{Q}(u^\varepsilon) + c_s^2 \nabla_x \rho^\varepsilon = \quad -\mathbb{Q}\left(\nabla_x \cdot \left(\sum_{i=0}^n w_i A_i \mathcal{F}_i^\varepsilon\right)\right) - \varepsilon \mathbb{Q}(\nabla_x \cdot (u^\varepsilon \otimes u^\varepsilon)) - \varepsilon \mathbb{Q}(\mathcal{R}_T^\varepsilon),
\end{cases}
\tag{73}
$$

where the second equation is obtained by simply applying the projection $\mathbb{Q}$ to equation (69). Combining corollary 9 with proposition 13, we observe that $\{\rho^\varepsilon\}_\varepsilon$ and $\{\mathbb{Q}(u^\varepsilon)\}_\varepsilon$ are both uniformly bounded in $L^\infty([0, T_0]; H^m(\mathbf{R}_x^d))$. By proposition 13 and the local energy inequality (44), we have that

$$
\int_0^{T_0} \left\| \mathbb{Q}\left(\nabla_x \cdot \left(\sum_{i=0}^n w_i A_i \mathcal{F}_i^\varepsilon\right)\right) \right\|_{H^{m-1}(\mathbf{R}_x^d)} dt \lesssim T_0^{\frac{1}{2}} \left(\int_0^{T_0} \|\mathcal{F}^\varepsilon(t)\|_{H_x^m L_{v,w}^2}^2 \, dt\right)^{\frac{1}{2}}
$$
$$
\lesssim \varepsilon (2+\nu)^{\frac{1}{2}} \|g_0\|_{H_x^m L_{v,w}^2}.
$$

Combining propositions 4 and 13 and the local energy inequality (44), we can deduce that

$$
\int_0^{T_0} \left\| \mathbb{Q}(\nabla_x \cdot (u^\varepsilon \otimes u^\varepsilon)) \right\|_{H^{m-1}(\mathbf{R}_x^d)} dt \lesssim \int_0^{T_0} \|u^\varepsilon(t)\|_{H^m(\mathbf{R}_x^d)}^2 \, dt
$$
$$
\lesssim T_0 (2+\nu) \|g_0\|_{H_x^m L_{v,w}^2}^2.
\tag{74}
$$

Moreover, by the Plancherel's identity, we observe that the $L_t^1 H_x^{m-1}$ norm of $\mathbb{Q}(\mathcal{R}_T^\varepsilon)$ follows estimate (74) as well. Therefore, we show that the right hand side of the second equation in (73) converges to zero in the strong sense in $L^1([0, T_0]; H^{m-1}(\mathbf{R}_x^d))$ as $\varepsilon \to 0$. By a compensated compactness result due to Lions and Masmoudi [21] (see also [10, theorem A.2]), which basically says that fast oscillating acoustic waves do not contribute to the macroscopic dynamics in the incompressible limits, we can conclude that

$$
\mathbb{P}(\nabla_x \cdot (\mathbb{Q}(u^\varepsilon) \otimes \mathbb{Q}(u^\varepsilon))) \to 0
$$

in the sense of distributions as $\varepsilon \to 0$. This completes the proof of lemma 16. □

Lemma 17. *For any* $\Phi \in C^\infty([0, T_0]; C_{c,\sigma}^\infty(\mathbf{R}_x^d))$, *it holds that*

$$
\int_0^{T_0} \int_{\mathbf{R}^d} \mathbb{P}(\nabla_x \cdot (u^\varepsilon \otimes u^\varepsilon)) \cdot \Phi \, dx \, dt \to \int_0^{T_0} \int_{\mathbf{R}^d} (\nabla_x \cdot (u \otimes u)) \cdot \Phi \, dx \, dt
$$

as $\varepsilon \to 0$.

Proof. We decompose

$$
\int_0^{T_0} \int_{\mathbf{R}^d} (\nabla_x \cdot (\mathbb{P}(u^\varepsilon) \otimes \mathbb{P}(u^\varepsilon))) \cdot \Phi \, dx \, dt
$$
$$
= \int_0^{T_0} \int_{\mathbf{R}^d} (\mathbb{P}(u^\varepsilon) - u) \cdot \nabla_x \Phi \cdot \mathbb{P}(u^\varepsilon) \, dx \, dt + \int_0^{T_0} \int_{\mathbf{R}^d} u \cdot \nabla_x \Phi \cdot (\mathbb{P}(u^\varepsilon) - u) \, dx \, dt
$$
$$
+ \int_0^{T_0} \int_{\mathbf{R}^d} (\nabla_x \cdot (u \otimes u)) \cdot \Phi \, dx \, dt.
$$

Combining lemma 15 with proposition 13, corollary 9 and remark 10, we deduce that

$$\left| \int_0^{T_0} \int_{\mathbf{R}^d} (\mathbb{P}(u^\varepsilon) - u) \cdot \nabla_x \Phi \cdot \mathbb{P}(u^\varepsilon) \, \mathrm{d}x \, \mathrm{d}t \right| + \left| \int_0^{T_0} \int_{\mathbf{R}^d} u \cdot \nabla_x \Phi \cdot (\mathbb{P}(u^\varepsilon) - u) \, \mathrm{d}x \, \mathrm{d}t \right|$$

$$\lesssim T_0 (2 + \nu)^{\frac{1}{2}} \|\mathbb{P}(u^\varepsilon) - u\|_{L^\infty \left([0,T_0]; L^2(\mathbf{R}_x^d)\right)} \|g_0\|_{L^\infty \left([0,T_0]; H_x^m L_{v,w}^2\right)} \|\nabla_x \Phi\|_{L^\infty \left([0,T_0]; L^\infty(\mathbf{R}_x^d)\right)}$$

$$\to 0 \quad \text{as} \quad \varepsilon \to 0.$$

The convergence of the remainder $\mathcal{R}_{\mathbb{P}}(u^\varepsilon)$ to zero in the sense of distributions is guaranteed by lemma 16. We thus obtain lemma 17. $\qquad\square$

Summarising all the convergence results that we have derived in this paper, we have proved that

$$\int_0^{T_0} \int_{\mathbf{R}^d} \left\{ \partial_t \mathbb{P}(u^\varepsilon) - c_s^2 \nu \Delta_x \mathbb{P}(u^\varepsilon) + \mathbb{P}(\nabla_x \cdot (u^\varepsilon \otimes u^\varepsilon)) \right\} \cdot \Phi \, \mathrm{d}x \, \mathrm{d}t$$

$$\to \int_{\mathbf{R}^d} u_0 \cdot \Phi(x,0) \, \mathrm{d}x - \left\{ \int_0^{T_0} \int_{\mathbf{R}^d} u \cdot \partial_t \Phi + (u \otimes u) : \nabla_x \Phi - c_s^2 \nu u \cdot \Delta_x \Phi \, \mathrm{d}x \, \mathrm{d}t \right\}$$

as $\varepsilon \to 0$ for any $\Phi(x,t) \in C_c^\infty([0,T_0]; C_{c,\sigma}^\infty(\mathbf{R}_x^d))$, i.e.

$$u \in C\left([0,T_0]; H^{m-1}(\mathbf{R}_x^d)\right) \cap L^\infty\left([0,T_0]; H^m(\mathbf{R}_x^d)\right)$$

is a local weak solution to the incompressible Navier–Stokes equations

$$\begin{cases} \partial_t u - c_s^2 \nu \Delta_x u + \nabla_x \cdot (u \otimes u) + \nabla_x p = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad \nabla_x \cdot u = 0 \end{cases}$$

with initial data $u(x,0) = \mathbb{P}(u_0)$.

5. Characterisation for isotropic lattices

The purpose of this section is to characterise the 2D and 3D isotropic lattices associated with the speed of sound $c_s = 3^{-\frac{1}{2}}$ under a restriction condition on the particle velocity set $\mathcal{V}$.

5.1. Relation to the cubature formula

Given a lattice $(\mathcal{V}, w)$, by considering $\mathbf{n}_0 := \{0, 1, 2, \ldots, n\}$ as the state space, we can define a corresponding discrete and finite measure space $(\mathbf{n}_0, 2^{\mathbf{n}_0}, P)$ and measurable functions X_α ($\alpha \in \{1, 2, \ldots, d\}$) by setting

$$P(\{i\}) := w_i \quad \text{and} \quad X_\alpha(i) := v_{i,\alpha} \quad (\forall\, i \in \mathbf{n}_0).$$

On the other hand, suppose that we are given a measure space $(\Omega, 2^\Omega, P)$ with Ω being a finite set and the probability measure P having full support, i.e., this means that $P(\{y_i\}) > 0$ for each $y_i \in \Omega$, and measurable functions X_α ($\alpha \in \{1, 2, \ldots, d\}$) on this measure space. Then, we can construct a lattice by setting

$$v_{i\alpha} := X_\alpha(y_i) \quad \text{and} \quad w_i := P(\{y_i\}) \quad (\forall\, y_i \in \Omega)$$

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and $v_i := (v_{i1}, v_{i,2}, \ldots, v_{id})$. Hence, there is a one-to-one mapping between lattices and finite measure spaces with full support combined with d measurable functions on these measure spaces. Let $c_s > 0$ and $Y_\alpha := c_s X_\alpha$ for any α. Then, the definition of an isotropic lattice associated with the speed of sound c_s can be rephrased as the following.

Definition 18. A lattice $(\mathcal{V}, w)$ is isotropic with the speed of sound $c_s > 0$ if the following conditions hold for its corresponding measure space $(\Omega, 2^\Omega, P)$ and the measurable functions Y_α ($\alpha \in \{1, 2, \ldots, d\}$):

1. $(\Omega, 2^\Omega, P)$ is a probability space.
2. $E[Y_\alpha] = 0$ for any $1 \leqslant \alpha \leqslant d$.
3. $E[Y_\alpha Y_\beta] = \delta_{\alpha\beta}$ for any $1 \leqslant \alpha, \beta \leqslant d$.
4. $E[Y_\alpha Y_\beta Y_\gamma] = 0$ for any $1 \leqslant \alpha, \beta, \gamma \leqslant d$.
5. $E[Y_\alpha Y_\beta Y_\gamma Y_\zeta] = \delta_{\alpha\beta}\delta_{\gamma\zeta} + \delta_{\alpha\gamma}\delta_{\beta\zeta} + \delta_{\alpha\zeta}\delta_{\beta\gamma}$ for any $1 \leqslant \alpha, \beta, \gamma, \zeta \leqslant d$.

Here $E[\cdot]$ represents the expectation of '$\cdot$', i.e. the integral of '$\cdot$' with respect to measure P.

Let Z_α ($\alpha \in \{1, 2, \ldots, d\}$) be independent Gaussian random variables with $E[Z_\alpha] = 0$ and $E[Z_\alpha^2] = 1$ for $\alpha \in \{1, 2, \ldots, d\}$. Then, conditions for $\mathbb{Y} = (Y_1, Y_2, \ldots, Y_d)$ in definition 18 are summarised as

$$E[f(\mathbb{Y})] = E[f(\mathbb{Z})]$$

for any polynomial f of d variables with degree less than or equal to 4 where $\mathbb{Z} = (Z_1, Z_2, \ldots, Z_d)$. To find such $\mathbb{Y} = (Y_1, Y_2, \ldots, Y_d)$ taking a finite number of values is a well-studied problem in the context of cubature formulas, see e.g. [25]. For example,

- For $d = 2$, there is a solution with $n = 7$, i.e. the D2Q7 scheme exists,
- For $d = 3$, there is a solution with $n = 13$, i.e. the D3Q13 scheme exists,
- For $d = 4$, there is a solution with $n = 22$,
- For $d = 5$, there is a solution with $n = 33$,
- For $d = 6$, there is a solution with $n = 44$,
- For $d = 7$, there is a solution with $n = 57$,
- For $d \geqslant 8$, there is a solution with $n = d^2 + 3d + 3$.

5.2. 2D and 3D isotropic lattices

In this section, we always consider $c_s = 3^{-\frac{1}{2}}$. Without this causing of any ambiguity, when we refer to isotropic lattices, we omit the words 'associated with the speed of sound $3^{-\frac{1}{2}}$'.

Proposition 19. *Suppose that the lattice $(\mathcal{V}, w)$ is isotropic and the support of X_α is in $[-1, 1]$ for any $\alpha \in \{1, 2, \ldots, d\}$. Then,*

$$P(X_\alpha = 0) = \frac{2}{3}, \quad P(X_\alpha = 1) = P(X_\alpha = -1) = \frac{1}{6}$$

for any $\alpha \in \{1, 2, \ldots, d\}$.

Proof. Fix $\alpha \in \{1, 2, \ldots, d\}$. Since the support of X_α is in $[-1, 1]$, it certainly holds that $X_\alpha^2 \geqslant X_\alpha^4$, i.e. $P(X_\alpha^2 \geqslant X_\alpha^4) = 1$. Since the algebraic conditions for an isotropic lattice say that $E[X_\alpha^2] = E[X_\alpha^4] = \frac{1}{3}$, we must have $P(X_\alpha^2 > X_\alpha^4) = 0$, i.e. $P(X_\alpha^2 = X_\alpha^4) = 1$. That means that $P(X_\alpha \in \{-1, 0, 1\}) = 1$. Then, $E[X_\alpha^2] = \frac{1}{3}$ implies that $P(X_\alpha = 0) = \frac{2}{3}$ and $E[X_\alpha] = 0$ implies that $P(X_\alpha = 1) = P(X_\alpha = -1) = \frac{1}{6}$. $\qquad\square$

Proposition 20. *Suppose that the lattice* $(\mathcal{V}, w)$ *is isotropic and the support of* X_α *is in* $[-1, 1]$ *for any* $\alpha \in \{1, 2, \ldots, d\}$. *Then, for* $\alpha, \beta \in \{1, 2, \ldots, d\}$ *satisfying* $\alpha \neq \beta$, *it holds that*

- $P(X_\alpha = X_\beta = 0) = \frac{4}{9}$,
- $P(X_\alpha = \sigma_\alpha, X_\beta = 0) = \frac{1}{9}$ *for* $\sigma_\alpha \in \{-1, 1\}$,
- $P(X_\alpha = \sigma_\alpha, X_\beta = \sigma_\beta) = \frac{1}{36}$ *for* $\sigma_\alpha, \sigma_\beta \in \{-1, 1\}$.

Proof. Let $\alpha, \beta \in \{1, 2, \ldots, d\}$ with $\alpha \neq \beta$. For any $k, j \in \{-1, 0, 1\}$, for simplicity of notations we denote $P(X_\alpha = k, X_\beta = j)$ by $p_{k,j}$. Note that $E[X_\alpha X_\beta] = 0$ implies that

$$p_{1,1} + p_{-1,-1} - p_{1,-1} - p_{-1,1} = 0, \tag{75}$$

$E[X_\alpha^2 X_\beta] = 0$ implies that

$$p_{1,1} - p_{-1,-1} - p_{1,-1} + p_{-1,1} = 0 \tag{76}$$

and $E[X_\alpha^2 X_\beta^2] = \frac{1}{9}$ implies that

$$p_{1,1} + p_{-1,-1} + p_{1,-1} + p_{-1,1} = \frac{1}{9}. \tag{77}$$

Adding (75) and (76) together, we obtain that $p_{1,1} = p_{1,-1}$ and $p_{-1,-1} = p_{-1,1}$. Adding (75) and (77) together, we obtain that

$$p_{1,1} + p_{-1,-1} = \frac{1}{18} = p_{1,-1} + p_{-1,1}. \tag{78}$$

Moreover, we have that

$$P(X_\alpha = 1) = \frac{1}{6} = p_{1,1} + p_{1,0} + p_{1,-1}, \quad P(X_\alpha = -1) = \frac{1}{6} = p_{-1,1} + p_{-1,0} + p_{-1,-1}. \tag{79}$$

By adding the two equations in (79) together and making use of (78), we deduce that

$$p_{1,0} + p_{-1,0} = \frac{2}{9}. \tag{80}$$

By proposition 19, we have that $\frac{2}{3} = P(X_\beta = 0) = p_{1,0} + p_{0,0} + p_{-1,0}$. Hence, equality (80) implies that $p_{0,0} = \frac{4}{9}$. In addition, by the linearity of expectation, we can deduce from $E[X_\alpha + X_\beta] = 0$ that

$$2p_{1,1} + p_{1,0} + p_{0,1} = p_{0,-1} + p_{-1,0} + 2p_{-1,-1} \tag{81}$$

and from $E[X_\alpha - X_\beta] = 0$ that

$$p_{1,0} + 2p_{1,-1} + p_{0,-1} = p_{0,1} + 2p_{-1,1} + p_{-1,0}. \tag{82}$$

Subtracting (82) from (81) and noting that $p_{1,1} = p_{1,-1}$ and $p_{-1,-1} = p_{-1,1}$, we deduce that $p_{0,1} = p_{0,-1}$. Using proposition 19 again, we have that $\frac{2}{3} = P(X_\alpha = 0) = p_{0,1} + p_{0,0} + p_{0,-1}$, i.e. it holds that

$$p_{0,1} = p_{0,-1} = \frac{1}{9}. \tag{83}$$

Since (83) holds for any $\alpha, \beta \in \{1, 2, \ldots, d\}$ satisfying $\alpha \neq \beta$, by interchanging values of α and β, we deduce that $p_{1,0} = p_{-1,0} = \frac{1}{9}$. Finally, by substituting values of $p_{1,0}$ and $p_{-1,0}$ back into (79), we obtain that

$$p_{1,1} = p_{1,-1} = \frac{1}{36} = p_{-1,-1} = p_{-1,1}.$$

This completes the proof of proposition 20. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 21. *Let $d = 2$ and $(\mathcal{V}, w)$ be a lattice. Suppose that the support of X_α is in $[-1, 1]$ for any $\alpha = 1, 2$. Then, the lattice $(\mathcal{V}, w)$ is isotropic if and only if the scheme is D2Q9.*

Proof. Sufficiency is an outcome of propositions 19 and 20. Necessity is a well-known fact which can be easily checked by direct calculations, see e.g. [16]. $\qquad\qquad\qquad\square$

Proposition 22. *Suppose that the lattice $(\mathcal{V}, w)$ is isotropic and the support of X_α is in $[-1, 1]$ for any $\alpha \in \{1, \ldots, d\}$. Suppose that $\alpha, \beta, \gamma \in \{1, \ldots, d\}$ satisfying $\alpha \neq \beta$, $\alpha \neq \gamma$ and $\beta \neq \gamma$. Then, there exist constants $c \in [0, \frac{1}{72}]$ and c_α, c_β, $c_\gamma \in [-\frac{1}{72}, \frac{1}{72}]$ such that*

- $P(X_\alpha = X_\beta = X_\gamma = 0) = \frac{1}{3} - 8c$,
- $P(X_\alpha = \sigma_\alpha, X_\beta = X_\gamma = 0) = \frac{1}{18} + 4c + 4\sigma_\alpha c_\alpha$　*for*　$\sigma_\alpha \in \{-1, 1\}$,
- $P(X_\alpha = \sigma_\alpha, X_\beta = \sigma_\beta, X_\gamma = 0) = \frac{1}{36} - 2c - 2\sigma_\alpha c_\alpha - 2\sigma_\beta c_\beta$　*for*　$\sigma_\alpha, \sigma_\beta \in \{-1, 1\}$,
- $P(X_\alpha = \sigma_\alpha, X_\beta = \sigma_\beta, X_\gamma = \sigma_\gamma) = c + \sigma_\alpha c_\alpha + \sigma_\beta c_\beta + \sigma_\gamma c_\gamma$　*for*　$\sigma_\alpha, \sigma_\beta, \sigma_\gamma \in \{-1, 1\}$.

Moreover, the constants $c, c_\alpha, c_\beta, c_\gamma$ satisfy the inequalities

$$\begin{cases} |c_\alpha| + |c_\beta| + |c_\gamma| \leqslant & c, \\ c + |c_\zeta| + |c_\eta| \leqslant & \frac{1}{72} \quad \text{for any} \quad \zeta, \eta \in \{\alpha, \beta, \gamma\} \quad \text{with} \quad \zeta \neq \eta. \end{cases}$$

Proof. Let $\alpha, \beta, \gamma \in \{1, 2, \ldots, d\}$ satisfy the relations $\alpha \neq \beta$, $\beta \neq \gamma$ and $\alpha \neq \gamma$. For any $i, j, k \in \{-1, 0, 1\}$, for simplicity of notations we denote $P(X_\alpha = i, X_\beta = j, X_\gamma = k)$ by $p_{i,j,k}$. Since $E[X_\alpha X_\beta X_\gamma] = 0$ and $E[X_\alpha^2 X_\beta X_\gamma] = 0$, it holds that

$$0 = E\left[\left(X_\alpha + X_\alpha^2\right) X_\beta X_\gamma\right] = 2p_{1,1,1} - 2p_{1,1,-1} - 2p_{1,-1,1} + 2p_{1,-1,-1} \qquad (84)$$

and

$$0 = E\left[\left(X_\alpha - X_\alpha^2\right) X_\beta X_\gamma\right] = -2p_{-1,1,1} + 2p_{-1,1,-1} + 2p_{-1,-1,1} - 2p_{-1,-1,-1}. \qquad (85)$$

By (84), we have that

$$p_{1,1,1} - p_{1,1,-1} = p_{1,-1,1} - p_{1,-1,-1}. \qquad (86)$$

Replacing (α, β) by (β, α), we deduce from (86) that

$$\begin{aligned} P(X_\alpha = 1, X_\beta = -1, X_\gamma = 1) &- P(X_\alpha = 1, X_\beta = -1, X_\gamma = -1) \\ &= P(X_\alpha = 1, X_\beta = 1, X_\gamma = 1) - P(X_\alpha = 1, X_\beta = 1, X_\gamma = -1) \\ &= P(X_\beta = 1, X_\alpha = 1, X_\gamma = 1) - P(X_\beta = 1, X_\alpha = 1, X_\gamma = -1) \\ &= P(X_\beta = 1, X_\alpha = -1, X_\gamma = 1) - P(X_\beta = 1, X_\alpha = -1, X_\gamma = -1), \end{aligned} \qquad (87)$$

i.e. we obtain that $p_{1,-1,1} - p_{1,-1,-1} = p_{-1,1,1} - p_{-1,1,-1}$. In addition, (85) implies that

$$p_{-1,1,1} - p_{-1,1,-1} = p_{-1,-1,1} - p_{-1,-1,-1}. \tag{88}$$

Combining (86)–(88), we can conclude that the quantity $p_{\sigma_\alpha,\sigma_\beta,1} - p_{\sigma_\alpha,\sigma_\beta,-1}$ is independent of $\sigma_\alpha, \sigma_\beta \in \{-1,1\}$. Hence, without causing any ambiguity, we may set $2c_\gamma := p_{\sigma_\alpha,\sigma_\beta,1} - p_{\sigma_\alpha,\sigma_\beta,-1}$ for $\sigma_\alpha, \sigma_\beta \in \{-1,1\}$. By applying analogous argument to consider

$$
\begin{aligned}
E\left[X_\alpha\left(X_\beta + X_\beta^2\right)X_\gamma\right] &= 0 = E\left[X_\alpha\left(X_\beta - X_\beta^2\right)X_\gamma\right], \\
E\left[X_\alpha X_\beta\left(X_\gamma + X_\gamma^2\right)\right] &= 0 = E\left[X_\alpha X_\beta\left(X_\gamma - X_\gamma^2\right)\right],
\end{aligned}
$$

we can deduce that $p_{1,\sigma_\beta,\sigma_\gamma} - p_{-1,\sigma_\beta,\sigma_\gamma} =: 2c_\alpha$ is independent of $\sigma_\beta, \sigma_\gamma \in \{-1,1\}$ and $p_{\sigma_\alpha,1,\sigma_\gamma} - p_{\sigma_\alpha,-1,\sigma_\gamma} =: 2c_\beta$ is independent of $\sigma_\alpha, \sigma_\gamma \in \{-1,1\}$.

Let $c := p_{1,1,1} - c_\alpha - c_\beta - c_\gamma$. Then,

$$
\begin{aligned}
p_{1,1,-1} &= p_{1,1,1} - 2c_\gamma \implies p_{1,1,-1} = c + c_\alpha + c_\beta - c_\gamma, \\
p_{-1,1,1} &= p_{1,1,1} - 2c_\alpha \implies p_{-1,1,1} = c - c_\alpha + c_\beta + c_\gamma, \\
p_{1,-1,1} &= p_{1,1,1} - 2c_\beta \implies p_{1,-1,1} = c + c_\alpha - c_\beta + c_\gamma.
\end{aligned}
\tag{89}
$$

Using (89), we further deduce that

$$
\begin{aligned}
p_{-1,1,-1} &= p_{-1,1,1} - 2c_\gamma \implies p_{-1,1,-1} = c - c_\alpha + c_\beta - c_\gamma, \\
p_{1,-1,-1} &= p_{1,-1,1} - 2c_\gamma \implies p_{1,-1,-1} = c + c_\alpha - c_\beta - c_\gamma, \\
p_{-1,-1,1} &= p_{-1,1,1} - 2c_\beta \implies p_{-1,-1,1} = c - c_\alpha - c_\beta + c_\gamma, \\
p_{-1,-1,-1} &= p_{-1,-1,1} - 2c_\gamma \implies p_{-1,-1,-1} = c - c_\alpha - c_\beta - c_\gamma.
\end{aligned}
$$

Summarising what we have obtained, it holds that

$$p_{\sigma_\alpha,\sigma_\beta,\sigma_\gamma} = c + \sigma_\alpha c_\alpha + \sigma_\beta c_\beta + \sigma_\gamma c_\gamma \tag{90}$$

for any $\sigma_\alpha, \sigma_\beta, \sigma_\gamma \in \{-1,1\}$ with $c := p_{1,1,1} - c_\alpha - c_\beta - c_\gamma$.

By the fourth bullet point of proposition 20 and (90), we can deduce that for $\sigma_\beta, \sigma_\gamma \in \{-1,1\}$

$$p_{0,\sigma_\beta,\sigma_\gamma} = p_{\sigma_\beta,\sigma_\gamma} - p_{1,\sigma_\beta,\sigma_\gamma} - p_{-1,\sigma_\beta,\sigma_\gamma} = \frac{1}{36} - 2c - 2\sigma_\beta c_\beta - 2\sigma_\gamma c_\gamma.$$

Analogously, for $\sigma_\alpha, \sigma_\gamma \in \{-1,1\}$, it holds that

$$p_{\sigma_\alpha,0,\sigma_\gamma} = \frac{1}{36} - 2c - 2\sigma_\alpha c_\alpha - 2\sigma_\gamma c_\gamma \tag{91}$$

and for $\sigma_\alpha, \sigma_\beta \in \{-1,1\}$, it holds that

$$p_{\sigma_\alpha,\sigma_\beta,0} = \frac{1}{36} - 2c - 2\sigma_\alpha c_\alpha - 2\sigma_\beta c_\beta.$$

Furthermore, for $\sigma_\gamma \in \{-1,1\}$, we deduce from the third bullet point of proposition 20 and (91) that

$$p_{0,0,\sigma_\gamma} = p_{0,\sigma_\gamma} - p_{1,0,\sigma_\gamma} - p_{-1,0,\sigma_\gamma} = \frac{1}{18} + 4c + 4\sigma_\gamma c_\gamma. \tag{92}$$

Similarly, for $\sigma_\alpha \in \{-1, 1\}$, it holds that

$$p_{\sigma_\alpha, 0, 0} = \frac{1}{18} + 4c + 4\sigma_\alpha c_\alpha$$

and for $\sigma_\beta \in \{-1, 1\}$, it holds that

$$p_{0, \sigma_\beta, 0} = \frac{1}{18} + 4c + 4\sigma_\beta c_\beta.$$

Using the first bullet point of proposition 20 and (92), we deduce that

$$p_{0,0,0} = p_{0,0} - p_{0,0,1} - p_{0,0,-1} = \frac{1}{3} - 8c.$$

Finally, we let

$$\widetilde{\text{sgn}}(x) = \begin{cases} 1, & \text{if } x \geqslant 0, \\ -1, & \text{if } x < 0. \end{cases}$$

By considering (90), we can deduce from

$$P(X_\alpha = -\widetilde{\text{sgn}}(c_\alpha), X_\beta = -\widetilde{\text{sgn}}(c_\beta), X_\gamma = -\widetilde{\text{sgn}}(c_\gamma)) \geqslant 0$$

that

$$0 \leqslant |c_\alpha| + |c_\beta| + |c_\gamma| \leqslant c.$$

Since $P(X_\alpha = \widetilde{\text{sgn}}(c_\alpha), X_\beta = \widetilde{\text{sgn}}(c_\beta), X_\gamma = 0) \geqslant 0$, we have that

$$c + |c_\alpha| + |c_\beta| \leqslant \frac{1}{72}.$$

Similarly, by considering $P(X_\alpha = \widetilde{\text{sgn}}(c_\alpha), X_\beta = 0, X_\gamma = \widetilde{\text{sgn}}(c_\gamma)) \geqslant 0$ and $P(X_\alpha = 0, X_\beta = \widetilde{\text{sgn}}(c_\beta), X_\gamma = \widetilde{\text{sgn}}(c_\gamma)) \geqslant 0$, we also have that

$$c + |c_\alpha| + |c_\gamma| \leqslant \frac{1}{72} \quad \text{and} \quad c + |c_\beta| + |c_\gamma| \leqslant \frac{1}{72}.$$

This completes the proof of proposition 22. $\qquad\square$

Proposition 23. *Let $d = 3$ and $(\mathcal{V}, w)$ be a lattice. Suppose that the support of X_α is in $\{-1, 0, 1\}$ for any $\alpha = 1, 2, 3$. For any $1 \leqslant \alpha, \beta, \gamma \leqslant 3$ satisfying $\alpha \neq \beta$, $\alpha \neq \gamma$ and $\beta \neq \gamma$, if there exist constants $c \in [0, \frac{1}{72}]$ and c_α, c_β, $c_\gamma \in [-\frac{1}{72}, \frac{1}{72}]$ such that*

- $P(X_\alpha = 0) = \frac{2}{3}$ *and* $P(X_\alpha = 1) = P(X_\alpha = -1) = \frac{1}{6}$,
- $P(X_\alpha = X_\beta = 0) = \frac{4}{9}$,
- $P(X_\alpha = \sigma_\alpha, X_\beta = 0) = \frac{1}{9}$ *for* $\sigma_\alpha \in \{-1, 1\}$,
- $P(X_\alpha = \sigma_\alpha, X_\beta = \sigma_\beta) = \frac{1}{36}$ *for* $\sigma_\alpha, \sigma_\beta \in \{-1, 1\}$,
- $P(X_\alpha = X_\beta = X_\gamma = 0) = \frac{1}{3} - 8c$,
- $P(X_\alpha = \sigma_\alpha, X_\beta = X_\gamma = 0) = \frac{1}{18} + 4c + 4\sigma_\alpha c_\alpha$ *for* $\sigma_\alpha \in \{-1, 1\}$,

- $P(X_\alpha = \sigma_\alpha, X_\beta = \sigma_\beta, X_\gamma = 0) = \frac{1}{36} - 2c - 2\sigma_\alpha c_\alpha - 2\sigma_\beta c_\beta$ *for* $\sigma_\alpha, \sigma_\beta \in \{-1,1\}$,
- $P(X_\alpha = \sigma_\alpha, X_\beta = \sigma_\beta, X_\gamma = \sigma_\gamma) = c + \sigma_\alpha c_\alpha + \sigma_\beta c_\beta + \sigma_\gamma c_\gamma$ *for* $\sigma_\alpha, \sigma_\beta, \sigma_\gamma \in \{-1,1\}$,

then $(\mathcal{V}, w)$ *is isotropic.*

Proof. This proof contains nothing but direct calculations. It can be easily verified that

$$
\begin{cases}
E[X_\alpha] = 0, \\[4pt]
E[X_\alpha^2] = \dfrac{1}{3}, \\[4pt]
E[X_\alpha^3] = 0, \\[4pt]
E[X_\alpha^4] = \dfrac{1}{3}, \\[4pt]
\displaystyle\sum_{\sigma_\alpha \in \{-1,0,1\}} P(X_\alpha = \sigma_\alpha) = 1, \\[10pt]
\displaystyle\sum_{\sigma_\alpha, \sigma_\beta \in \{-1,0,1\}} P(X_\alpha = \sigma_\alpha, X_\beta = \sigma_\beta) = 1, \\[10pt]
\displaystyle\sum_{\sigma_\alpha, \sigma_\beta, \sigma_\gamma \in \{-1,0,1\}} P(X_\alpha = \sigma_\alpha, X_\beta = \sigma_\beta, X_\gamma = \sigma_\gamma) = 1.
\end{cases}
$$

For $\alpha, \beta \in \{1,2,3\}$ with $\alpha \neq \beta$, we have that

$$
\begin{cases}
E[X_\alpha X_\beta] = \dfrac{1}{36} \cdot \{1^2 + 1 \cdot (-1) + (-1) \cdot 1 + (-1)^2\} = 0, \\[8pt]
E[X_\alpha^2 X_\beta] = \dfrac{1}{36} \cdot \{1^3 + 1^2 \cdot (-1) + (-1)^2 \cdot 1 + (-1)^2 \cdot (-1)\} = 0, \\[8pt]
E[X_\alpha^3 X_\beta] = \dfrac{1}{36} \cdot \{1^4 + 1^3 \cdot (-1) + (-1)^3 \cdot 1 + (-1)^3 \cdot (-1)\} = 0, \\[8pt]
E[X_\alpha^2 X_\beta^2] = \dfrac{1}{36} \cdot \{1^4 + 1^2 \cdot (-1)^2 + (-1)^2 \cdot 1^2 + (-1)^2 \cdot (-1)^2\} = \dfrac{1}{9}.
\end{cases}
$$

For $\alpha, \beta, \gamma \in \{1,2,3\}$ with $\alpha \neq \beta$, $\alpha \neq \gamma$ and $\beta \neq \gamma$, we have that

$$
\begin{aligned}
E[X_\alpha X_\beta X_\gamma] &= 1^3 \cdot (c + \sigma_\alpha + \sigma_\beta + \sigma_\gamma) + 1^2 \cdot (-1) \cdot (c + \sigma_\alpha + \sigma_\beta - \sigma_\gamma) \\
&\quad + 1 \cdot (-1) \cdot 1 \cdot (c + \sigma_\alpha - \sigma_\beta + \sigma_\gamma) + 1 \cdot (-1) \cdot (-1) \cdot (c + \sigma_\alpha - \sigma_\beta - \sigma_\gamma) \\
&\quad + (-1) \cdot 1^2 \cdot (c - \sigma_\alpha + \sigma_\beta + \sigma_\gamma) + (-1) \cdot 1 \cdot (-1) \cdot (c - \sigma_\alpha + \sigma_\beta - \sigma_\gamma) \\
&\quad + (-1) \cdot (-1) \cdot 1 \cdot (c - \sigma_\alpha - \sigma_\beta + \sigma_\gamma) + (-1) \cdot (-1) \cdot (-1) \cdot (c - \sigma_\alpha - \sigma_\beta - \sigma_\gamma) \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
E[X_\alpha^2 X_\beta X_\gamma] &= 1^4 \cdot (c + \sigma_\alpha + \sigma_\beta + \sigma_\gamma) + 1^3 \cdot (-1) \cdot (c + \sigma_\alpha + \sigma_\beta - \sigma_\gamma) \\
&\quad + 1^2 \cdot (-1) \cdot 1 \cdot (c + \sigma_\alpha - \sigma_\beta + \sigma_\gamma) + 1^2 \cdot (-1) \cdot (-1) \cdot (c + \sigma_\alpha - \sigma_\beta - \sigma_\gamma) \\
&\quad + (-1)^2 \cdot 1^2 \cdot (c - \sigma_\alpha + \sigma_\beta + \sigma_\gamma) + (-1)^2 \cdot 1 \cdot (-1) \cdot (c - \sigma_\alpha + \sigma_\beta - \sigma_\gamma) \\
&\quad + (-1)^2 \cdot (-1) \cdot 1 \cdot (c - \sigma_\alpha - \sigma_\beta + \sigma_\gamma) + (-1)^2 \cdot (-1) \cdot (-1) \cdot (c - \sigma_\alpha - \sigma_\beta - \sigma_\gamma) \\
&= 0.
\end{aligned}
$$

We thus obtain proposition 23. $\qquad\square$

Remark 24. Combining propositions 22 and 23, we obtain a characterisation for 3D isotropic lattices. Different from the 2D case, 3D isotropic lattice associated with the speed of sound $3^{-\frac{1}{2}}$ is not unique even if we require X_α to be supported in $[-1,1]$. For example,

- If $c = \frac{1}{72}$ and $c_\alpha = c_\beta = c_\gamma = 0$, this corresponds to the D3Q15 scheme,
- If $c = c_\alpha = c_\beta = c_\gamma = 0$, this corresponds to the D3Q19 scheme,
- If $c = \frac{1}{216}$ and $c_\alpha = c_\beta = c_\gamma = 0$, this corresponds to the D3Q27 scheme,

see e.g. [16].

Remark 25. The characterisations of 2D and 3D isotropic lattices (corollary 21, propositions 22 and 23, respectively) in this section rely sharply on the assumption $c_{\mathrm{s}} = 3^{-\frac{1}{2}}$. Indeed, suppose that $c_{\mathrm{s}} \neq 3^{-\frac{1}{2}}$. Then, even if we assume that X_α is supported in $[-1,1]$, it is possible for X_α to be supported on values other than $-1, 0, 1$. For example, in the 2D case, if we consider $\mathcal{V} = \{v_0, v_1, v_2, \ldots, v_6\} \subset \mathbf{R}^2$ where

$$\begin{cases} v_0 = 0, \\ v_j = \left(\cos \dfrac{2\pi j}{6}, \sin \dfrac{2\pi j}{6} \right), \quad j = 1, \ldots, 6 \end{cases}$$

and $w = (w_0, w_1, w_2, \ldots, w_6) \in \mathbf{R}_+^7$ where

$$\begin{cases} w_0 = \dfrac{1}{2}, \\ w_j = \dfrac{1}{12}, \quad j = 1, \ldots, 6, \end{cases}$$

then it can be easily checked that the combination $(\mathcal{V}, w)$ is an isotropic lattice associated with the speed of sound $c_{\mathrm{s}} = \frac{1}{2}$, which corresponds to the D2Q7 scheme. If we apply the proof of proposition 19 to the setting of $(\mathcal{V}, w)$, then we shall face the problem that $E[X_\alpha^2] \neq E[X_\alpha^4]$, which further induces the possibility that $P(X_\alpha^2 > X_\alpha^4) > 0$. Hence, it is not necessary for X_α to support in $\{-1, 0, 1\}$.

Remark 26. It is worth noting that requiring the random variable X_α to be supported in $[-1,1]$ is not necessary for a lattice to be isotropic. If $(\mathcal{V}, w)$ is an isotropic lattice associated with the speed of sound c_{s}, then for any real number $\lambda > 0$, the lattice $(\lambda\mathcal{V}, w)$ is also an isotropic lattice associated with speed of sound $\frac{c_{\mathrm{s}}}{\lambda}$ where $\lambda\mathcal{V} := \{\lambda v_i \mid 0 \leqslant i \leqslant n\}$.

6. Numerical investigation of the hydrodynamic limit of the DVBGK Boltzmann problem

In this section we conduct numerical computations designed to illustrate the hydrodynamic limit investigated in section 4. This is done by comparing the macroscopic velocity fields derived from the solutions of the DVBGK Boltzmann system (5) which are obtained for a sequence of decreasing values of ε to the solutions of the Navier–Stokes system (17) where the initial condition for the former system, $g^{0,\varepsilon}$, is chosen to be consistent with the initial condition for the latter, u_0, cf theorem 3. For simplicity and to fix attention, we focus here on the 2D setting ($d = 2$) and consider both the DVBGK Boltzmann and the Navier–Stokes systems (5) and (17) on a torus $\mathbf{T}_x^2 := [0,1]^2$ rather than on an unbounded domain $\mathbf{R}_x^2$, which is motivated by computational considerations. Defining vorticity $\omega := \nabla_x^\perp \cdot u$, where $\nabla_x^\perp := (-\partial_{x_2}, \partial_{x_1})$, it

is convenient to rewrite the Navier–Stokes system in the vorticity form obtained applying the operator $\left(\nabla_x^\perp \cdot\right)$ to the first equation in (17)

$$\begin{cases} \partial_t\omega + (u \cdot \nabla_x)\,\omega - c_s^2 \nu \Delta_x \omega = 0, \\ \qquad\qquad\qquad\qquad u = \nabla_x^\perp \Delta_x^{-1} \omega, \\ \qquad\qquad\qquad \omega\,(x,0) = \omega_0 := \nabla_x^\perp \cdot u_0, \end{cases} \tag{93}$$

where Δ_x^{-1} is the inverse Laplacian equipped with the periodic boundary conditions on $\mathbf{T}_x^2$. Since solutions of the 2D Navier–Stokes system (93) conserve the mean vorticity, $(\mathrm{d}/\mathrm{d}t) \int_{\mathbf{T}_x^2} \omega(x,t)\,\mathrm{d}x = 0$, $t \geqslant 0$, it is necessary to restrict the admissible initial conditions ω_0 to have a zero mean, $\int_{\mathbf{T}_x^2} \omega_0(x)\,\mathrm{d}x = 0$, as this will ensure the Laplacian can be inverted at all times (cf the second equation in (93)). For the DVBGK Boltzmann problem we consider the D2Q9 lattice given by (9), (10) and the speed of sound $c_s = \frac{1}{\sqrt{3}}$.

We then focus on the following two macroscopic initial conditions which for convenience are expressed here in the vorticity form:

- the Taylor–Green vortex

$$\omega_0\,(x_1, x_2) = 10 \sin\,(2\pi a x_1) \sin\,(2\pi b x_2), \tag{94}$$

where $a = b = 2$, which leads to a solution of system (93) where the nonlinear term $(u \cdot \nabla_x)\omega$ vanishes identically for all times $t \geqslant 0$, such that ω effectively solves the heat Equation $\partial_t \omega - c_s^2 \nu \Delta_x \omega = 0$ on $\mathbf{T}_x^2$ and has therefore the form [32]

$$\omega\,(x_1, x_2, t) = \omega_0\,(x_1, x_2)\,\exp\left[-4\pi^2\left(a^2 + b^2\right) c_s^2 \nu t\right], \tag{95}$$

- the perturbed Taylor–Green vortex

$$\omega_0\,(x_1, x_2) = -\sin\,(2\pi x_1) \sin\,(2\pi x_2) + \exp\left[-50\left((x_1 - 1/2)^2 + (x_2 - 1/2)^2\right)\right] + C, \quad (96)$$

where the constant C is chosen to ensure that this initial data satisfies the zero-mean condition; solutions of the Navier–Stokes system (93) subject to the initial condition (96) need to be found numerically as described below in section 6.1; since our spatial domain is the 2D torus, the function in (96) should be interpreted as a periodic extension of a function restricted to $0 \leqslant x_1, x_2 \leqslant 1$ and since the second term on the right-hand side is not a periodic function, this initial condition has in fact discontinuous derivatives; however, due to the rapid decay of the exponential function away from the point $(1/2, 1/2)$, the magnitude of this discontinuity is negligible and does not affect the numerical results.

The microscopic initial condition $g^{0,\varepsilon}$ for the DVBGK Boltzmann system is then obtained from (94) or (96) using relation (16) in which we set $u_0 = \nabla_x^\perp \Delta_x^{-1} \omega_0$ and, without the loss of generality, $\rho_0(x_1, x_2) = 1$, $\forall(x_1, x_2) \in \mathbf{T}_x^2$.

6.1. Numerical solution of the DVBGK Boltzmann and Navier–Stokes equations

The DVBGK Boltzmann and Navier–Stokes systems (5) and (93) are approximated numerically using a standard pseudo-spectral approach where the dependence of the solution on the space variable x is represented in terms of a truncated Fourier series and the use of this ansatz

can be interpreted as application of the cutoff function Λ_ε introduced in (11). Derivatives are then evaluated exactly in the Fourier (spectral) space whereas all product terms are computed in the physical space with de-aliasing [7]. Discrete Fourier transforms relating the two representations are evaluated using the FFT algorithm. The system of coupled ordinary differential equations resulting from this approximation is then integrated in time using the standard fourth-order Runge–Kutta method (RK4). This is an explicit approach and hence only conditionally stable, so care must be exercised to ensure the stability of these computations by using a sufficiently small time step Δt. In fact, the DVBGK Boltzmann problem is 'stiff', in the sense that the maximum allowable time step decreases as $\varepsilon \to 0$. For both problems introduced above, the stability and accuracy of computations was carefully verified by performing these computations with different time steps Δt and different numerical resolutions N (where N is the number of grid points used to discretise the domain $\mathbf{T}_x^2$ in each direction). Moreover, the numerical solution of the Navier–Stokes system (93) can be validated by considering the problem with the initial condition (94), where an exact solution is available, cf (95). This is in fact a demanding test as it requires the vanishing of the nonlinear term in the approximate solution. This approach has been numerically implemented in MATLAB and the code is available on Github [23].

In the numerical solutions of the DVBGK Boltzmann and the Navier–Stokes system (5) and (93) discussed below the spatial resolution is $N = 128$. In the latter case the time step is $\Delta t = 2 \times 10^{-6}$, whereas in the former it is in the range $\Delta t \in [2 \times 10^{-6}, 2 \times 10^{-4}]$ depending on the value of ε (Δt is smaller for decreasing ε). Due to this limitation, we did not consider values of ε smaller than 10^{-1}.

6.2. Results

Before examining the dependence of the difference between the macroscopic solutions $\omega^\varepsilon := \nabla_x^\perp \cdot u^\varepsilon$ of the DVBGK Boltzmann system (5) and the solutions ω of the Navier–Stokes system (93) on ε, we provide some additional information about the flows considered. In both cases, the kinematic viscosity (or the relaxation time in case of the DVBGK Boltzmann system) is $\nu = 10^{-4}$.

As is evident from its exact solution in (95), due to the absence of the nonlinear effects, the structure of the Navier–Stokes flow corresponding to the initial condition (94) remains unchanged in time with only its magnitude vanishing exponentially. Snapshots of the vorticity field $\omega(x,t)$ at times $t = 0, 8, 16, 32$ during the evolution of the Navier–Stokes flow with the initial condition (96) are shown in figure 1. Stretching of the vortices present in the initial field ω_0 is evident, giving rise to the formation of thin elongated filaments which is the main dynamic mechanism sustaining the enstrophy cascade in 2D turbulence [19]. To further characterise this flow, we define the enstrophy and palinstrophy as

$$\mathcal{E}(t) := \frac{1}{2} \int_{\mathbf{T}_x^2} \omega^2 \, \mathrm{d}x, \tag{97}$$

$$\mathcal{P}(t) := \frac{1}{2} \int_{\mathbf{T}_x^2} |\nabla \omega|^2 \, \mathrm{d}x, \tag{98}$$

which are equivalent to, respectively, the L^2 norm and the H^1 seminorm of the vorticity. It can be shown that, for smooth solution of the Navier–Stokes system (93), the time evolution of these quantities is governed by the equations [2]

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = -2\nu\mathcal{P}, \tag{99}$$
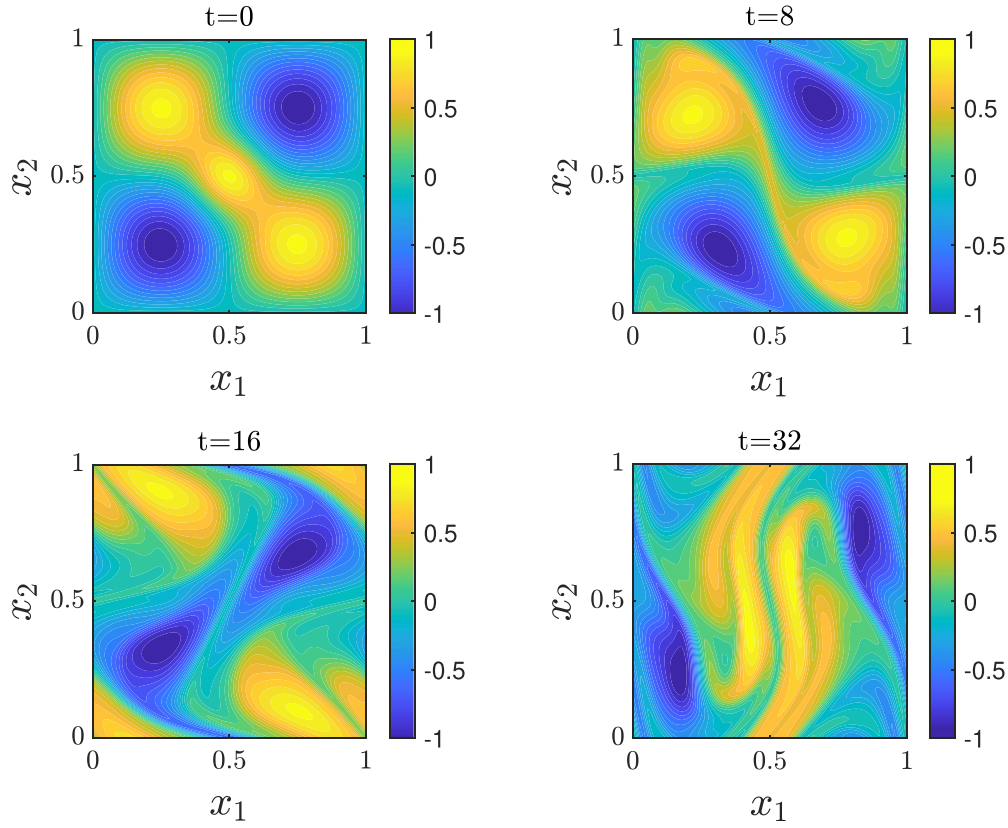
Figure 1. Snapshots of the vorticity field $\omega(x_1, x_2, t)$ at times $t = 0, 8, 16, 32$ during the evolution of the Navier–Stokes flow with the initial condition (96).

$$\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t} = \int_{\mathbf{T}_x^2} (u \cdot \nabla_x) \omega \Delta_x \omega \, \mathrm{d}x - \nu \int_{\mathbf{T}_x^2} (\Delta_x \omega)^2 \, \mathrm{d}x. \tag{100}$$

We see that, while the enstrophy $\mathcal{E}(t)$ is a nonincreasing function of time, the evolution of the palinstrophy is a result of the competition between the stretching of vorticity filaments and viscous dissipation represented, respectively, by the cubic and the negative-definite quadratic term in (100). This behaviour is indeed confirmed by the data shown in figures 2(a) and (b), where, in particular, we observe that the palinstrophy $\mathcal{P}(t)$ initially increases by a factor of about three. This demonstrates that the Navier–Stokes flow corresponding to the initial condition (96) is at least for the times considered in figures 1 and 2 dominated by nonlinear effects. An animated version of these figures is included as the supplemental data of this article.

Finally, our main results are shown in figure 3 where we plot the (normalised) norms of the difference between the macroscopic vorticity ω^ε in the DVBGK Boltzmann solution and the vorticity ω in the Navier–Stokes flow at a certain time $t = T$ as functions of ε. The time when this difference is evaluated is chosen as $T = 1$ and $T = 32$ for the problems with the initial conditions (94) and (96). This latter time corresponds to an instance shortly after the palinstrophy $\mathcal{P}(t)$ has peaked, cf figure 2(b). It is clear from figure 3 that the norm of the difference can in both cases be represented very accurately by a power-law relation obtained
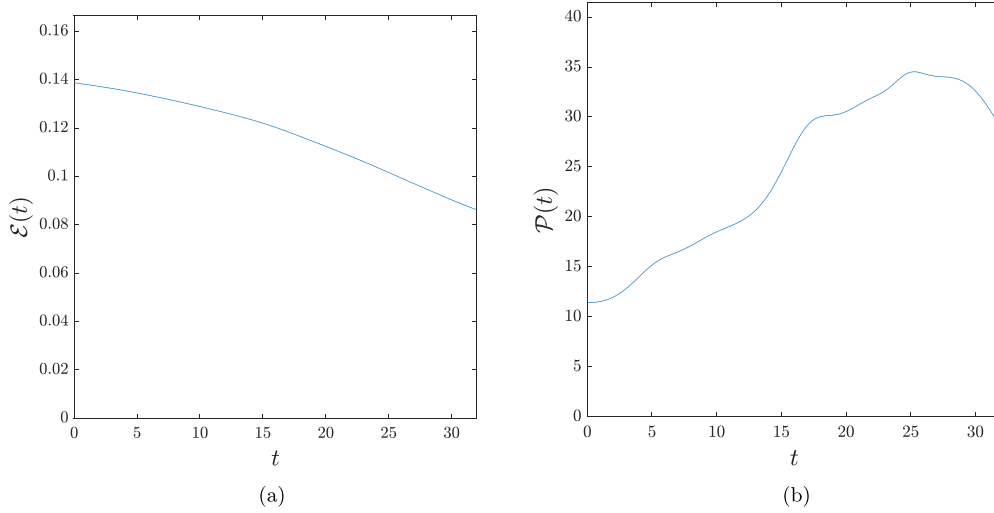
Figure 2. The time evolution of (a) the enstrophy $\mathcal{E}(t)$ and (b) the palinstrophy $\mathcal{P}(t)$, cf (97) and (98), in the Navier–Stokes flow with the initial condition (96).

via a least-squared fit

$$\frac{\|\omega\left(T\right) - \omega^{\varepsilon}\left(T\right)\|_{L^2\left(\mathbf{T}_x^2\right)}}{\|\omega\left(T\right)\|_{L^2\left(\mathbf{T}_x^2\right)}} \approx \begin{cases} 1.0458 \times 10^{-6}\,\varepsilon^{2.0022} & \text{for the flow with initial condition (94)} \\ 7.2178 \times 10^{-4}\,\varepsilon^{2.0001} & \text{for the flow with initial condition (96)} \end{cases}$$
$$= \mathcal{O}\left(\varepsilon^2\right). \tag{101}$$

Essentially the same power-law behaviour (except for a different prefactor) was also observed in both cases for different values of T. We note that in the light of the identity $\|\nabla_x^{\perp} \cdot u\|_{L^2\left(\mathbf{T}_x^2\right)} = \|\nabla_x u\|_{L^2\left(\mathbf{T}_x^2\right)}$, relation (101) provides an indication about the rate with which the hydrodynamic limit is achieved in 2D in terms of the H^1 norm (at the level of velocity).

7. Summary and conclusions

In conclusion, in this paper we establish that a local classical solution to the d-dimensional incompressible Navier–Stokes equations ($d \geqslant 2$) can be constructed by considering the hydrodynamic limit of a discrete-velocity Boltzmann equation with a quadratic polynomial-type collision operator obtained as a simplification of the BGK Boltzmann collision operator, as long as the finite set of particle velocities and their weights satisfy five summation conditions that characterise symmetries for lattices.

The case for space dimension $d = 1$ is somehow tricky. Since the Helmholtz projection is not available in 1D, it is hard to eliminate the term $\partial_x \rho^{\varepsilon}$ which arises in the conservation of momentum, i.e. the first equation of (57), and is of order $\mathcal{O}(\varepsilon^{-1})$. On the other hand, from the conservation of mass, i.e. the second equation of (57), it always holds that the limiting macroscopic fluid velocity u is independent of the space variable x. Hence, our main theorem excludes the 1D case.

In addition, we characterise the 2D and 3D lattice structures that lead to the incompressible Navier–Stokes equations in the hydrodynamic limit when the speed of sound is $c_{\mathrm{s}} = 3^{-\frac{1}{2}}$.
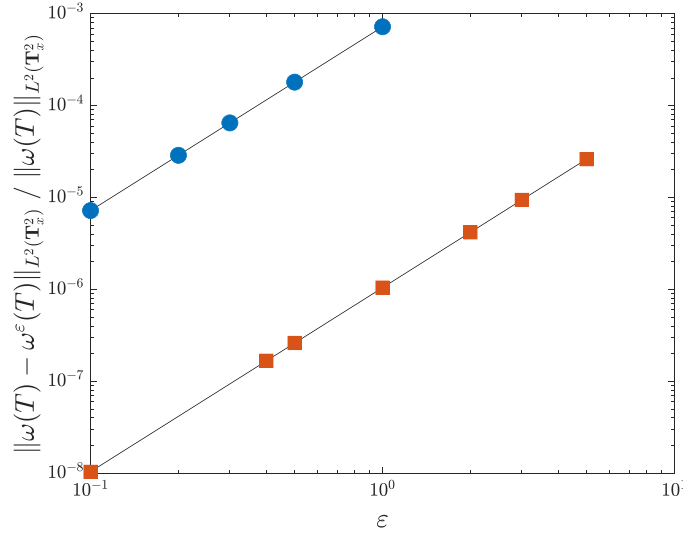
Figure 3. Dependence of $\|\omega(T) - \omega^\varepsilon(T)\|_{L^2(\mathbf{T}_x^2)}/\|\omega(T)\|_{L^2(\mathbf{T}_x^2)}$ on ε for the DVBGK Boltzmann and Navier–Stokes flows with the initial conditions (94) (red squares) and (96) (blue circles). Solid lines represent the power-law fits (101).

Furthermore, by implementing the D2Q9 scheme numerically, we observe that the hydrodynamic limit of the DVBGK Boltzmann equation is attained at the rate $\mathcal{O}(\varepsilon^2)$, see figure 3. This convergence rate agrees with the theory of Junk-Yong [15] in which the hydrodynamic limit of the same type of Boltzmann equation was rigorously justified based on the Hilbert expansion.

However, the strict convergence of the entire sequence $\{u^\varepsilon\}_\varepsilon$ is hard to justify within the framework of the BGL program. Although we have a strong convergence for $\{\mathbb{P}(u^\varepsilon)\}_\varepsilon$ (cf lemma 15), this strong convergence is guaranteed by suppressing subsequences due to the application of the Arzelà–Ascoli theorem. Besides, the Arzelà–Ascoli theorem does not provide an explicit convergence rate. In addition, due to the presence of a high frequency term in (57) whose coefficient is of order $\mathcal{O}(\varepsilon^{-1})$, we have to overcome the problem that $\{\mathbb{Q}(u^\varepsilon)\}_\varepsilon$ does not have enough compactness properties to converge strongly. Therefore, if one works with the BGL program to establish the hydrodynamic limit, it is a challenging task to rigorously prove the rate of convergence of the limiting sequence.

Finally, we would like to raise some open questions that deserve further consideration. Considering the BGL program, it is interesting to ask:

- can the rate of convergence for $\{u^\varepsilon\}_\varepsilon$ be proved without considering the Hilbert expansion?

Considering the work of Junk and Yong [15], it is also interesting to pose the question:

- can the Hilbert expansion approach be made to work in higher dimensions (i.e. $d \geqslant 3$) and for more general lattice structures in the DVBGK Boltzmann equation (5)?

Furthermore, in realistic applications one would need to consider the motion of fluid in a domain with a solid boundary. Hence, considering the DVBGK Boltzmann equation in such a domain, it is worth asking:

- how can the boundary conditions be designed such that proper fluid boundary conditions are obtained in hydrodynamic limit?
- does the Hilbert expansion approach work when effects of boundaries are introduced?

In many applications of the LBM to high-Reynolds number flows, adaptive relaxation times are used for higher-order moments to stabilise the numerical scheme. To mimic this, it is possible to introduce a space-time dependent relaxation time to the DVBGK Boltzmann model. In this case, the solvability of the DVBGK Boltzmann equation should be guaranteed provided the inverse of the space-time dependent relaxation time can be controlled, e.g. $\nu(x,t)^{-1} \in L^\infty$. In the formal derivation of the Navier–Stokes equations, as long as the kinematic viscosity of the fluid is independent of space and time, then we expect that the rigorous proof should be analogous to the analysis presented in the present paper. Since detailed calculations are needed to confirm this, we leave it as a problem that deserves future considerations:

- does analogous analysis hold if a space-time dependent relaxation time is considered in the DVBGK Boltzmann equation (5)?

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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