LAST (Family) NAME: SOLUTIONS **Test # 1 – Summer 2022**

FIRST NAME: SOLUTIONS **Math 2C03 – Pritpal Matharu**

ID # : SOLUTIONS **July 13th, 2022 — Duration: 75 Minutes**

Instructions: This exam consists of 5 questions in 7 pages. Indicate your answers clearly in the appropriate places. Justify your answers in order to receive full credit. No books, notes or calculators allowed. The last page is a formula sheet for your convenience, which is the only page you may remove.

GOOD LUCK!

FOR MARKING ONLY

#	Mark	(out of)
1.	10	**10 points**
2.	14	**14 points**
3.	12	**12 points**
4.	6	**6 points**
5.	8	**8 points**
Total	50	**50 points**

Continued. . .

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1. [**10 points**]

(a) Solve the IVP

$$\begin{cases} y'(x) = y^2 - 1, \\ y(x=0) = 0. \end{cases}$$

Write your final solution in explicit form, $y(x) = \ldots$

Noting that we have a separable equation:

$$\frac{dy}{dx} = y^2 - 1$$

$$\int \frac{1}{y^2-1}\,dy = 1\,dx \qquad \to \text{use partial fractions}$$

$$\int \frac{-\frac{1}{2}}{y+1} + \frac{\frac{1}{2}}{y-1}\,dy = 1\,dx \qquad \to \text{perform integration}$$

$$-\frac{1}{2}\ln|y+1| + \frac{1}{2}\ln|y-1| = x + C_1$$

$$\ln\left|\frac{y-1}{y+1}\right| = 2x + C_2 \qquad \to C_2 = 2C_1$$

$$\left|\frac{y-1}{y+1}\right| = e^{2x+C_2}$$

$$\frac{y-1}{y+1} = Ae^{2x} \qquad \to A = \pm e^{C_2}$$

$$y - 1 = (y+1)\,Ae^{2x}$$

$$y(x) = \frac{Ae^{2x}+1}{1-Ae^{2x}} \qquad \to \text{use IC: } y(0) = 0$$

$$0 = \frac{A+1}{1-A}$$

$$A = -1$$

The solution to the IVP is: $\boxed{y(x) = \frac{1-e^{2x}}{1+e^{2x}}}$ *or* $\boxed{= \frac{e^{-2x}-1}{e^{-2x}+1}}$ [6 *points*]

(b) Consider the same equation as above but now with initial condition $\boxed{y(0) = 1}$.
True or False: there exists a unique solution with this initial condition in some interval I. If True, find the unique solution $y(x)$. If False, find two different solutions $y_1(x)$ and $y_2(x)$.

True. $\boxed{y(x) = 1}$ *is the unique solution, by the Local Existence and Uniqueness Theorem (Overview_2022_06_22.pdf / Theorem 1.2.1 Existence of a Unique Solution in Zill)* [4 *points*]

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2. [**14 points**] Find $a \in \mathbb{R}$ such that the solution $x(t)$ of

$$\begin{cases} x'' + x' - 2x = 0, \\ x(0) \ = a, \\ x'(0) = 1, \end{cases}$$

tends to zero as $t \to \infty$.

First, we solve this linear homogeneous constant coefficient ODE with the ansatz $x(t) = e^{mt}$:

$$\begin{aligned} x'' + x' - 2x &= 0 && \to x(t) = e^{mt} \\ m^2 e^{mt} + m e^{mt} - 2e^{mt} &= 0 \\ (m^2 + m - 2)e^{mt} &= 0 && \to e^{mt} \neq 0 \\ m^2 + m - 2 &= 0 \\ (m-1)\,(m+2) &= 0 \end{aligned}$$

Thus our roots are $m_1 = 1$ and $m_2 = -2$, which gives us a solution of

$$x(t) = c_1\, e^t + c_2\, e^{-2t}. \quad [4 \ points]$$

To ensure that the solution tends to zero as $t \to \infty$, we require $\underline{c_1 = 0}$. [4 *points*]
Thus,

$$\begin{aligned} x(t) &= c_2\, e^{-2t} && \to \text{use IC: } x(0)=a \\ a &= c_2\, e^0 \\ c_2 &= a && \to \text{use and take derivative of } x(t) \\ x'(t) &= -2c_2\, e^{-2t} && \to \text{use IC: } x'(0) = 1 \\ 1 &= -2c_2\, e^0 \\ c_2 = a &= -\frac{1}{2} \end{aligned}$$

Therefore, $\boxed{a = -\frac{1}{2}}$ [6 *points*]

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3. [**12 points**]

(a) Determine the interval I of existence and uniqueness for solutions of the initial value problem

$$\ln(x+3)\,y''(x) - (3-x^2)\,y'(x) + (x+5)\,y(x) = \frac{1}{16-x^2}, \quad y(0)=2, \quad y'(0)=-4$$

[Do not try to solve the equation, just give the interval!]

Let $a_2(x) = \ln(x+3)$, $a_1(x) = -(3-x^2)$, $a_0(x) = (x+5)$, and $g(x) = \frac{1}{16-x^2}$. By the Existence and Uniqueness Theorem - Linear Case (Lecture 2022_06_27 / Theorem 4.1.1 Existence of a Unique Solution in Zill), we must ensure that the initial point $x_0 = 0$ is contained in an interval I for which $a_2(x) \neq 0$ and $a_2(x)$, $a_1(x)$, $a_0(x)$, and $g(x)$ are continuous every $x \in I$.

$a_1(x)$ and $a_0(x)$ are continuous for every $x \in \mathbb{R}$, however $g(x)$ is discontinuous at $x = \pm 4$ and $a_2(x)$ is not defined for $x < -3$. In addition, $a_2(x) = 0$ if $x = -2$.

Since $x_0 = 0$ is contained between $x = -2$ and $x = 4$, $\boxed{I = (-2, 4)}$ [6 *points*]

(b) Verify that $y_1(x) = x^3$ and $y_2(x) = x^4$ form a fundamental solution set on the interval $I = (0, \infty)$ for

$$x^2y'' - 6xy' + 12y = 0.$$

To form a fundamental solution set, we first must ensure that the two solutions are indeed solutions.

Plugging y_1 into the ODE:

$$x^2(3\cdot 2\cdot x) - 6x(3\cdot x^2) + 12(x^3) = 0$$
$$6x^3 - 18x^3 + 12x^3 = 0$$
$$0 = 0 \qquad \rightarrow LHS = RHS \text{ [2 point]}$$

Similarly for y_2:

$$x^2(4\cdot 3\cdot x^2) - 6x(4\cdot x^3) + 12(x^4) = 0$$
$$12x^4 - 24x^4 + 12(x^4) = 0$$
$$0 = 0 \qquad \rightarrow LHS = RHS \text{ [2 point]}$$

Now, we must ensure that $y_1(x)$ and $y_2(x)$ are linearly independent. Since we know they are indeed solutions, then from the Theorem for Linearly Independent Solutions (Lecture 2022_06_27 / Theorem 4.1.3 Theorem for the Criterion for Linearly Independent Solutions in Zill) we know that the solutions are linearly independent if and only if their Wronskian is not equal to zero for every $x \in I$. Thus, computing the Wronskian

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^3 & x^4 \\ 3x^2 & 4x^3 \end{vmatrix} = 4x^6 - 3x^6 = x^6 \neq 0 \quad \text{for } x \in (0,\infty).$$

Therefore, $y_1(x)$ and $y_2(x)$ form a fundamental solution set on the interval $I = (0, \infty)$ [2 *points*]

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4. [**6 points**] Find the general solution of the non-homogeneous equation

$$y'' - 2y' + y = \frac{e^x}{1+x^2}$$

Hint: $W = e^{2x}$ and you may use the formula sheet.

First, we solve the associated linear homogeneous constant coefficient ODE with the ansatz $y(x) = e^{mx}$:

$$\begin{aligned} y'' - 2y' + y &= 0 && \to y(x) = e^{mx} \\ m^2 e^{mx} - 2m e^{mx} + e^{mx} &= 0 \\ (m^2 - 2m + 1)e^{mx} &= 0 && \to e^{mx} \neq 0 \\ m^2 - 2m + 1 &= 0 \\ (m-1)(m-1) &= 0 \end{aligned}$$

Thus we have a repeated root $m_1 = m_2 = 1$. We have one solution $y_1(x) = e^x$ and to obtain a second linearly independent solution we use reduction of order, to get $y_2(x) = xe^x$. Thus our associated homogeneous solution is

$$y_H(x) = c_1 e^x + c_2 x e^x. \quad [3 \textit{ points}]$$

For the general solution $y(x) = y_H(x) + y_p(x)$, we also need to determine the particular solution. We need to use variation of parameters, where we assume $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ and determine the unknown functions $u_1(x)$ and $u_2(x)$. First we compute the Wronskian of the solutions W, W_1, and W_2 where we set $f(x) = \frac{e^x}{1+x^2}$ (equation is already in standard form, so we do not need to rearrange):

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = e^{2x}.$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & xe^x \\ \frac{e^x}{1+x^2} & xe^x + e^x \end{vmatrix} = -\frac{xe^{2x}}{1+x^2}.$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{1+x^2} \end{vmatrix} = \frac{e^{2x}}{1+x^2}.$$

Now, we can solve $u_1(x)$ and $u_2(x)$ by evaluating the integrals:

$$\begin{aligned} u_1(x) &= \int_{t=x} W_1/W\, dt = \int_{t=x} -\frac{te^{2t}}{(1+t^2)} \frac{1}{e^{2t}}\, dt = \int_{t=x} -\frac{t}{1+t^2}\, dt \qquad \to \textit{use substitution } v = 1+t^2 \\ &= -\frac{1}{2}\ln(x^2+1) \\ u_2(x) &= \int_{t=x} W_2/W\, dt = \int_{t=x} \frac{e^{2t}}{(1+t^2)} \frac{1}{e^{2t}}\, dt = \int_{t=x} \frac{1}{1+t^2}\, dt \qquad \to \textit{derivative of } \arctan \\ &= \tan^{-1}(x) \end{aligned}$$

Thus the particular solution is

$$\begin{aligned} y_p(x) &= u_1(x)y_1(x) + u_2(x)y_2(x) \\ &= \left(-\frac{1}{2}\ln(x^2+1)\right)(e^x) + \left(\tan^{-1}(x)\right)(xe^x)\,. \end{aligned}$$

Therefore the general solution is

$$\boxed{y(x) = y_H(x) + y_p(x) = c_1e^x + c_2xe^x - \frac{1}{2}\ln(x^2+1)\,e^x + \tan^{-1}(x)\,xe^x}.$$

[3 *points*]

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5. [**8 points**] Answer TRUE or FALSE to each of the following questions. Justify your answer (one sentence is fine).

(a) If $f_1(t), f_2(t)$ are two differentiable functions which are linearly independent on an interval I, then the Wronskian $W[f_1, f_2](t) \neq 0$ for all $t \in I$.

False. [1 *point*]
$f_1(t)$ and $f_2(t)$ must be solutions, not only functions for this to be true by Abel's theorem. For example, in lecture we showed that $f_1(t) = \sin(t)$ and $f_2(t) = t^2$ are linearly independent, however computing their Wronskian:

$$W(f_1, f_2)(t) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} \sin(t) & t^2 \\ \cos(t) & 2t \end{vmatrix} = 2t\sin(t) - t^2\cos(t).$$

Then $W(f_1, f_2)(t = \frac{\pi}{2}) = \pi \neq 0$, whereas $W(f_1, f_2)(t = 0) = 0$.
Thus $f_1(t) = \sin(t)$ and $f_2(t) = t^2$ are one possible counterexample to show this statement is false. [3 *points*]

(b) If $f(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ are both continuous for all $(x, y) \in \mathbb{R}^2$, then the solution $y(x)$ to the initial value problem $y'(x) = f(x, y),\ y(0) = y_0$ exists for every $x \in \mathbb{R}$.

False. [1 *point*]
This would only give us local existence of solutions, but not existence of solutions for every $x \in \mathbb{R}$. For example, $f(x, y) = e^{x+y}$ and $\frac{\partial f}{\partial y}(x, y) = e^{x+y}$ are both continuous in $\mathbb{R}^2$. However as shown in lecture, the solution with IC $y(0) = 0$ is $y(x) = -\ln(2 - e^x)$, which only exists for $x \in (-\infty, \ln(2))$.
Thus $f(x, y) = e^{x+y}$ is one possible counterexample to show this statement is false. [3 *points*]
If you wanted to make this statement true, you would require that $\frac{\partial f}{\partial y}(x, y)$ is also bounded (Global Existence and Uniqueness Theorem, which was not covered in this course).

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Formulas

$$\mu(t) = e^{-\int p(t)\,dt}$$

$$x_p = x_1 v_1 + x_2 v_2, \quad v_1 = -\int \frac{x_2(t) f(t)}{W}\,dt, \quad v_2 = \int \frac{x_1(t) f(t)}{W}\,dt$$

$$\frac{d}{du}(\cos u) = -\sin u, \quad \frac{d}{du}(\sin u) = \cos u$$

$$\frac{d}{du}(\ln u) = \frac{1}{u}, \quad \frac{d}{du}\left(\ln(1+u^2)\right) = \frac{2u}{1+u^2}$$

$$\frac{d}{du}(\arctan u) = \frac{1}{1+u^2}, \quad \frac{d}{du}(\arcsin u) = \frac{1}{\sqrt{1-u^2}}$$

$$\int u\,dv = u\,v - \int v\,du$$

END